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SUPERSYMMETRY - A FIELD THEORETICAL REVIEW

by

GEORGE TSOUROS

(C)

A THESIS

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Antkowiak
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Supervisor

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W. H. ...
.....

Date *May 5, 1989*

B. Roy
H. ...
...

I dedicate these pages
to my Guardian Angel,
impressing upon him
that I am only fooling
and warning him
to see to it that
there is no misunderstanding
when I go home.

Abstract

The inadequacies congenitally inherent in Grand Unified Theories - the most outstanding one being the gauge hierarchy problem - in combination with the Coleman - Mandula theorem, seem to imply that Supersymmetry and Supergravity constitute the only viable theories for unification within the framework of Quantum Field Theory. The group theoretical properties of Supersymmetry, are analyzed and exemplified in this project. Its most central part, examines the quantum field theoretical properties of the $N=1$ Supersymmetry without central charges (Wess - Zumino model), and the significant feature - characterizing all Supersymmetry models - of the absence of quadratic divergences to all orders in perturbation theory. Contrary to the original analysis of J. Iliopoulos and B. Zumino, the renormalization program of the aforementioned model in the present discussion, advances through the more conventional - though equivalent - explicit introduction of counterterms. Finally, some general remarks concerning the relevant phenomenology and present experimental status of Supersymmetry are as well furnished.

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If what ultimately renders this life meaningful is the struggle in it, and if this dissertation reflects my endeavour to that end, it would appear that it is my bounden duty to hereby express my gratitude to those who more or less contributed to it.

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throughout my work within the most unfavorable circumstances with which I was confronted.

As it would be highly arduous a task to mention all of them, I should perhaps make a symbolic reference to the greatest of them all: J. S. Bach!

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I. SUPERSYMMETRY AND UNIFICATION

A. Motivations for Supersymmetry

Unification of all fundamental interactions, in Nature, has been the "Holy Grail" of Theoretical Physics. The first successful attempt towards this direction was the unification of the electromagnetic and weak interactions [GLAS.80]. Today the Glashow - Weinberg - Salam theory, accurately describes the electroweak phenomena, and Quantum Chromodynamics (QCD) is accepted as the theory of the strong interactions. Thus all of low energy particle physics phenomena are describable in terms of what has come to be called: The Standard Model, a gauge theory based on $SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$.

There are several reasons why this gauge theory cannot be considered as the final theory. The most important is that it is not a unified theory at all. Each group factor comes with its own coupling strength. Even worse is the presence of the $U(1)$ factor, since $U(1)$ permits any number of coupling constants. In a non abelian group the coupling constant is fixed by the gauge boson self coupling and it must be the same for every matter multiplet except for Clebsh - Gordan coefficients. For $U(1)$ however, this is not so. In other words the present theory does not explain why the electric charge seems to be quantized and no particles with charge equal to πe are observed. For these reasons it would appear plausible to assume that $SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$ is

the remnant of a larger semi-simple group G which is spontaneously broken at very high energies, according to the scheme:

$$G \rightarrow \dots \overset{M}{\dots} \rightarrow SU_c(3) \otimes SU_L(2) \otimes U_Y(1) \xrightarrow{m_W} SU_L(2) \otimes U_Y(1) .$$

The groups G $SU(5)$, $SO(10)$, $E(6)$, constitute the obvious and natural step towards the unification of electroweak and strong interactions called justifiably Grand Unified Theories (GUTs) [NAN.80].

Gauge theories, contain two or possibly three independent "worlds". The world of radiation with the gauge bosons, the world of matter with the fermions and, finally - according to the present understanding - the world of Higgs scalars. In the framework of gauge theories, these worlds are essentially unrelated to each other. Given a group G , the world of radiation is completely determined, but there is no way of knowing a priori how many fermion representations should be introduced; the world of matter is to a great extent arbitrary. This arbitrariness is even more disturbing if one considers the world of Higgs scalars. Not only are their number and representations undetermined, but their mere presence introduces into the theory a large number of arbitrary parameters.

The situation becomes even more dramatic in Grand Unified theories where one may have to adjust parameters with as many as twenty six significant figures. This is the problem of gauge hierarchy which is connected to the two

enormously different mass scales at which spontaneous symmetry breaking occurs. The grand unification scale or in other words the scale at which the breaking of G to $SU(3) \otimes SU(2) \otimes U(1)$ occurs is: $M \sim O(10^{14}) \text{ GeV}$. This means that a certain Higgs field Φ acquires a non zero expectation value: $V = \langle \Phi \rangle \sim O(10^{14}) \text{ GeV}$. The second breaking, that of $SU_L(2) \otimes U_Y(1)$ occurs at 100 GeV , i.e. there should exist a second scalar field ϕ with $v = \langle \phi \rangle \sim O(10^2) \text{ GeV}$. But the combined Higgs potential will contain a term of the form: $\lambda \phi^2 \Phi^2$. Therefore, after the first breaking, the ϕ -mass will be expressed by [ILI.83]:

$$m_\phi^2 = \mu^2 + 2\lambda V^2$$

where μ is the mass appearing in the symmetric Lagrangian. On the other hand, it holds that: $V^2 \sim m_\phi^2$, so unless there is a very precise cancellation between μ^2 and $2\lambda V^2$, V^2 will turn out to be of order v^2 and the two breakings will come together, in other words the theory is not able to sustain naturally a gauge hierarchy. This fine tuning of parameters must be repeated order by order in perturbation theory, because unlike fermions, scalar field masses require quadratically divergent counterterms. The whole structure seems extremely unlikely.

It would therefore appear that Nature must have contrived a way to eliminate all this arbitrariness. One possibility which suggests itself is to connect the three worlds with some sort of symmetry principle. Then the

knowledge of the vector bosons will determine the fermions and the scalars, and the absence of quadratically divergent terms from the fermion masses will forbid their appearance in the scalar masses. Such a symmetry would obviously relate the fermion to the boson fields, and by definition, it would constitute what has come to be called: a SUPERSYMMETRY (SUSY).

There are even more reasons apart from the aforementioned gauge hierarchy problem, which provide good motivations for supersymmetric theories. Despite various shortcomings, Grand Unified Theories would at most unify different kinds of interactions, such as strong and electroweak, with each other and different kinds of matter, such as quarks and leptons with each other. The unification of "matter" with interaction is beyond the competence of GUTs. At the same time, the most obvious difference between say gravity and electrodynamics namely the absence of negative gravitational charges, can be shown to imply that only a supersymmetric theory can unify them [SOH.85]. Requiring that no deviation from standard quantum field theory, be deemed acceptable, a purely attractive interaction must be mediated by a field of even integer spin. The electromagnetic interaction on the other hand, which is not always positive, is mediated by a field with spin one. The Coleman Mandula theorem, [COL.67] forbids any direct symmetry transformations between fields of different integer spin and actually leaves supersymmetric theories as

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the only field theoretical models which might achieve unification of all interactions of Nature.

The Coleman - Mandula theorem implies that supersymmetry and its version which is covariant under general coordinate transformations - known as supergravity - are the only possibilities for unification within the framework of Quantum Field Theory. Failure of these theories to ultimately give results compatible with the real world, would imply that either unification or quantum field theory has to be dismissed.

However Quantum Field Theory allows for supersymmetry without any further assumptions, and it would be highly undesirable a situation to have to abandon either one of them. As a concluding remark it might be worth while to mention at this point, that the spin degree of freedom is not fundamentally understood or integrated into the gauge theory structure at present, while it would be with supersymmetry. Supersymmetric theories are mathematically better behaved. In addition, it would appear that eventually only supersymmetric field theories will be considered meaningful. But the supersymmetry of these theories need not be observable in the low energy effective theory of quarks leptons and gauge bosons, although it could be. Locally supersymmetric theories relate the generators of supersymmetric transformations to the generators of space-time transformations, so they couple the theory of gravity to the supersymmetric theory. It would appear, in

principle at least, that through this connection, gravity can eventually be unified with the strong and electroweak interactions.

II. THE SUPERSYMMETRY ALGEBRA

The generator of a symmetry is an operator in Hilbert space that replaces one incoming or outgoing multiparticle state with another and furthermore "leaves the physics intact" in the sense that it commutes with the S-matrix:

$$[S, G] = 0 \quad (II.1)$$

such an operator should act additively on direct products of states: This implies that it can be written as the product of an annihilation operator 'a', which annihilates a particle out of a multiparticle state, and a creation operator 'a[†]', which creates another particle in its stead with different internal quantum numbers and three momentum. The most general such product is:

$$G = \sum_{ij} \int d^3\vec{p} d^3\vec{q} a_i^\dagger(\vec{p}) K_{ij}(\vec{p}, \vec{q}) a_j(\vec{q}) \quad (II.2)$$

Using the symbol * for the convolutions in the previous relation, G can be expressed symbolically as:

$$G = a^\dagger * K * a \quad (II.2')$$

The operator G will replace some incoming quantum state |in> by another, G|in>. The fact that G commutes with the S-matrix implies that it makes no difference whether the state is "reshuffled" before or after an interaction has taken place: SG|in> = GS|in>. The sums in (II.2) run over all particle quantum numbers and thus also over all spin values.

There are terms which replace bosons by bosons, and others which replace fermions by fermions, bosons by fermions and fermions by bosons. Any G can be decomposed into an even part B and an odd part Q :

$$G = B + Q \quad (II.3)$$

where B contains only terms which replace bosons by bosons and fermions by fermions, while Q replaces bosons by fermions, and fermions by bosons. Symbolically it is expressed as:

$$B = b^\dagger * K_{bb} * b + f^\dagger * K_{ff} * f$$

$$Q = f^\dagger * K_{fb} * b + b^\dagger * K_{bf} * f \quad (II.4)$$

where b and f are annihilation operators for bosons and fermions respectively. The only assumption involved at this point, is the validity of the spin statistics theorem, and therefore the B 's can only change spin by integer amounts or not at all, whereas the Q 's must change the total spin of a state by one half, and are consequently supersymmetry generators.

The particle operators are required to obey the canonical quantization rules:

$$[b_i(\bar{p}), b_j^\dagger(\bar{q})] = \delta_{ij} \delta^3(\bar{p} - \bar{q}) \quad (II.5a)$$

$$\{f_i(\bar{p}), f_j^\dagger(\bar{q})\} = \delta_{ij} \delta^3(\bar{p} - \bar{q}) \quad (II.5b)$$

$$[b, b] = [b^\dagger, b^\dagger] = 0 \quad (II.5c)$$

$$\{f, f\} = \{f^\dagger, f^\dagger\} = 0 \quad (II.5d)$$

$$[b, f] = [b, f^\dagger] = [b^\dagger, f] = [b^\dagger, f^\dagger] = 0 \quad (II.5e)$$

Expression of these relations in a more elegant fashion, necessitates the introduction of the "graded commutator" symbol: $\{.....,.....\}$ which denotes the anticommutator if both operators are fermionic and the commutator in all other cases. The canonical quantization rules then read simply:

$$[a, a^\dagger] = 1, [a, a] = [a^\dagger, a^\dagger] = 0 \quad (II.5)$$

Having introduced the aforementioned operators, along with their quantization rules, the next step, involves the construction of a third symmetry generator G^3 from two known ones G^1 and G^2 . If both commute with the S-matrix then so does their product. The canonical quantization rules then suggest the commutator $[G^1, G^2]$.

This is to be applied first to two bosonic generators B^1 and B^2 . One can use the quantization rules along with the identities:

$$[ab, c] = a[b, c] + [a, c]b = a\{b, c\} - \{a, c\}b \quad (II.6)$$


to get:

$$[B^1, B^2] = B^3 \quad (\text{II.7a})$$

where the kernels of B^3 , which define it, are given by:

$$K_{bb}^3 = K_{bb}^1 * K_{bb}^2 - K_{bb}^2 * K_{bb}^1 ; K_{ff}^3 = K_{ff}^1 * K_{ff}^2 - K_{ff}^2 * K_{ff}^1 \quad (\text{II.7b})$$

This commutator is bilinear in particle operators. Consequently it is another generator of a symmetry. Likewise it can be shown that the commutator of a 'Q' with a 'B', gives another 'Q':

$$[Q^1, B^2] = Q^3 \quad (\text{II.8a})$$

with:

$$K_{fb}^3 = K_{fb}^1 * K_{bb}^2 - K_{ff}^2 * K_{fb}^1 ; K_{bf}^3 = K_{bf}^1 * K_{ff}^2 - K_{bb}^2 * K_{bf}^1 \quad (\text{II.8b})$$

However there is no way to decompose the commutator $[b^\dagger f, f^\dagger b]$, which appears in $[Q_1, Q_2]$ in such a way that only those proper combinations $\{ \dots, \dots \}$ appear which allow elimination of two of the four particle operators in the product. It is thus to be concluded that the commutator of two odd generators is not bilinear in particle operators and hence not a symmetry generator. However, it is found that the anticommutator of two Q's can be evaluated using the identities:

$$\{ab, c\} = a\{b, c\} - [a, c]b = a\{b, c\} + \{a, c\}b \quad (\text{II.9})$$

as well as the ones of eq.(II.6). The anticommutator of two Q 's turns out to be the generator of an even symmetry:

$$\{Q^1, Q^2\} = B^3 \quad (\text{II.10a})$$

with:

$$K_{bb}^3 = K_{bf}^1 * K_{fb}^2 + K_{bf}^2 * K_{fb}^1 ; K_{ff}^3 = K_{fb}^1 * K_{bf}^2 + K_{fb}^2 * K_{bf}^1 \quad (\text{II.10b})$$

The B 's and the Q 's satisfy the relations characteristic of a graded Lie algebra. It is the grading of the canonical quantization rules, i.e., that the fermions obey anticommutation relations, which induces a similar grading for the symmetry generators. The odd generators behave like fermionic objects, the even ones like bosonic objects.

So far the properties of a graded Lie algebra were introduced through relations (II.7 - II.10). Consideration of a basis (B_i, Q_a) in the corresponding Hilbert space, would render these formulae in their component form:

$$[B_i, B_j] = ic^k_{ij} B_k$$

$$[Q_a, B_i] = s^\beta_{ai} Q_\beta \quad (\text{II.11})$$

$$\{Q_a, Q_\beta\} = \gamma^i_{a\beta} B_i$$

The structure constants cannot be completely arbitrary. The c 's and the γ 's have symmetry properties:

$$c^k_{ij} = -c^k_{ji} ; \gamma^i_{\alpha\beta} = \gamma^i_{\beta\alpha} \quad (\text{II.12})$$

and all structure constants are subject to consistency conditions which follow from the graded Jacobi identities:

$$[[G^1, G^2], G^3] + \text{graded cyclic} = 0 \quad (\text{II.13})$$

The graded "cyclic sum" is defined just as the cyclic sum except that there is an additional minus - sign if two fermionic operators are interchanged, e.g. :

$$Q_a Q_\beta B_i + \text{graded cyclic} = Q_a Q_\beta B_i + B_i Q_a Q_\beta - Q_\beta B_i Q_a \quad (\text{II.14a})$$

The full set of graded Jacobi identities is then:

$$[[B_i, B_j], B_k] + [[B_k, B_i], B_j] + [[B_j, B_k], B_i] = 0$$

$$[[Q_a, B_i], B_j] + [[B_j, Q_a], B_i] + [[B_i, B_j], Q_a] = 0$$

(II.14b)

$$[\{Q_a, Q_\beta\}, B_i] + \{[B_i, Q_a], Q_\beta\} - \{[Q_\beta, B_i], Q_a\} = 0$$

$$[\{Q_a, Q_\beta\}, Q_\gamma] + \{[Q_\gamma, Q_a], Q_\beta\} + \{[Q_\beta, Q_\gamma], Q_a\} = 0$$

The requirement that these be fulfilled is equivalent to demanding that certain matrices constructed from the

structure constants, form a representation r of the algebra, the adjoint representation. These matrices are:

$$r(B_i) = \begin{bmatrix} C_i & 0 \\ 0 & S_i \end{bmatrix}, \quad r(Q_a) = \begin{bmatrix} 0 & \Sigma_a \\ \Gamma_a & 0 \end{bmatrix} \quad (II.15a)$$

with matrix elements:

$$(C_i)_j^k = ic_{ji}^k; \quad (S_i)_a^\beta = s_{ai}^\beta$$

$$(\Gamma_a)_\beta^i = \gamma_{ba}^i; \quad (\Sigma_a)_i^\beta = s_{ia}^\beta \quad (II.15b)$$

The requirement that these matrices form a representation is equivalent to the following four conditions:

1. the matrices C_i form a representation (the adjoint representation) of the bosonic Lie subalgebra which is spanned by the B 's alone;
2. the matrices S_i also form a representation of the Lie subalgebra (this representation need not be irreducible);
3. the γ_{β}^i are numerical invariants under the Lie subalgebra:

$$(S_i)_a^\delta \gamma_{\delta\beta}^j + (S_i)_\beta^\delta \gamma_{a\delta}^j - \gamma_{a\beta}^k (C_i)_k^j = 0 \quad (II.16)$$

4. there is a cyclic identity involving 's' and 'γ':

$$\gamma_{\alpha\beta}^i S_{\gamma i}^{\delta} + \text{cyclic} (\alpha \rightarrow \beta \rightarrow \gamma) = 0 \quad (\text{II.17})$$

Having discussed so far the canonical quantization rules, as well as the algebra of the supersymmetry generators, before proceeding to the supersymmetry algebra and its consequences, a brief analysis of the celebrated Coleman - Mandula theorem will be furnished.

The assumptions this theorem is based on are very general and can hardly be questioned from a physical point of view. The theorem, takes only Lie groups into consideration and consequently, fermionic generators and supersymmetry are excluded from its analysis. Nevertheless its results hold for the subset of all bosonic generators, and through the previous analysis, severely restrict the fermionic generators as well.

The Coleman - Mandula theorem states that on very general assumptions (non trivial S - matrix, etc.), any Lie group which contains the Poincare group P and an internal symmetry group G with generators T_s such that:

$$[T_r, T_s] = f_{rst} T_t$$

must be a direct product of P and G; or equivalently:

$$[P_a, T_s] = 0 = [M_{ab}, T_s]$$

In addition G must be of the form of a semisimple group with additional U(1) factors.

The bosonic generators are thus the four momenta and the six Lorentz generators, plus a certain number of hermitian internal symmetry generators. The algebra is that of the Poincare group:

$$[P_\mu, P_\nu] = 0$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho)$$

(II.18)

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho} M_{\mu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\rho} M_{\nu\sigma} + \eta_{\mu\sigma} M_{\nu\rho})$$

together with that of the internal symmetry group,

$$[B_r, B_s] = iC_{rs}^t B_t \quad (II.19)$$

The direct product manifests itself in the vanishing of the commutators:

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0 \quad (II.20)$$

which implies that the B_r must be translationally invariant Lorentz scalars.

The Casimir operators of the Poincare group are the mass-square operator $P^2 \equiv P_\mu P^\mu$, and the generalised spin operator $W^2 \equiv W_\mu W^\mu$, where W^μ is the Pauli Lubanski vector [RAM.81]:

$$W^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} \quad (II.21)$$

In the rest frame of a massive state it holds:

$$P_\mu = (m, 0, 0, 0) \quad \text{and} \quad W^2 = -m^2 L^2 \quad (\text{II.22a})$$

with:

$$L = (M_{23}, M_{31}, M_{12}) \quad (\text{II.22b})$$

These Casimir operators commute with the entire Poincare group and also with all internal symmetry generators:

$$[B_r, P^2] = 0 ; [B_r, W^2] = 0 \quad (\text{II.23})$$

The first of these relations implies that all members of an irreducible multiplet of the internal symmetry group must have the same mass (O'Raifeartaigh's theorem) [O'R.65]. In the same respect the second equation implies that they must also have the same spin. For massless states with discrete helicities it holds:

$$W_\mu = \lambda P_\mu \quad \text{with } (\lambda \text{ half-integer}), \quad (\text{II.24})$$

$$P^2 = W^2 = 0.$$

Again no B^r can change the helicity since $[B_r, P_\mu] = [B_r, W_\mu] = 0$, and the statement that internal symmetries cannot change spin still holds.

Stated in a positive way, the Coleman - Mandula theorem reads: All generators of supersymmetries must be fermionic, i.e., they must change the spin by a half-odd amount and

change the statistics of the state.

As a consequence, this no-go theorem leaves - as alternative - the exploration of the possibilities of the fermionic generators, the odd elements of the graded algebra [GOL.71] [VOL.73].

A positive metric on the relevant Hilbert space, is to be imposed as a first step:

$$\langle \dots | \{Q, Q^\dagger\} | \dots \rangle = |Q^\dagger | \dots \rangle|^2 + |Q | \dots \rangle|^2 > 0 \text{ if } : Q \neq 0 \quad (\text{II.25})$$

If there is no element of the bosonic subset of the generators that possesses all the properties required of $\{Q, Q^\dagger\}$, it shall be concluded that $Q=0$.

The commutator between M_{ab} and Q_{ai} , is to be investigated. As a result of (II.8a) it must be of the form: $[Q_{ai}, M_{ab}] = (b_{ab})_a^\beta Q_{\beta i}$, since by definition Q_{ai} are the only odd generators. The identity:

$$[[M_{ab}, M_{cd}], Q_{ai}] + [[Q_{ai}, M_{ab}], M_{cd}] + [[M_{cd}, Q_{ai}], M_{ab}] = 0$$

upon use of $[Q_{ai}, M_{cd}] = (b_{cd})_a^\beta Q_{\beta i}$ implies that:

$$[b_{ab}, b_{cd}]_a^\beta = -\delta_{ac} (b_{bd})_a^\beta - \delta_{bd} (b_{ac})_a^\beta + \delta_{ad} (b_{bc})_a^\beta + \delta_{bc} (b_{ad})_a^\beta$$

This means that the $(b_{cd})_a^\beta$ form a representation of the Lorentz algebra or equivalently, the Q_{ai} carry a representation of the Lorentz group. Q_{ai} will be selected to be in the $(0, \frac{1}{2}) \oplus (\frac{1}{2}, 0)$ representation of the Lorentz group

i.e.:

$$[Q_{ai}, M_{ab}] = \frac{1}{2}(\sigma_{ab})_a{}^\beta Q_{\beta a}. \quad (\text{Since } r(M_{ab}) = \frac{1}{2}\sigma_{ab} = \frac{1}{2}\sigma_c)$$

This then, leads to the following two relations:

$$[Q_{ai}, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_a{}^\beta Q_{\beta i} \quad (\text{II.26})$$

$$[\bar{Q}^i_a, M_{\mu\nu}] = -\frac{1}{2}\bar{Q}^i_\beta (\bar{\sigma}_{\mu\nu})^{\beta}{}_a$$

with the σ - matrices defined as in the Appendix.

The Index 'i' in Q_{ai} , labels all the different 2-spinors that are present in the theory and runs from 1 to some integer N. The assumption involved is that if Q is the generator of a symmetry then so is $Q^\dagger$. Since (Q_{ai}) must be in the complex conjugate representation, and thus be one of the $\bar{Q}$'s, they are numbered in such a way that always:

$$\bar{Q}^i_a = (Q_{ai})^\dagger \quad (\text{II.27})$$

Consideration of the $\{Q_a^i, Q_b^j\}$ anticommutator follows. This object must be composed of even generators and must be symmetric under interchange of $a \leftrightarrow b$ and $i \leftrightarrow j$. The even generators are those of the Poincare group, the internal symmetry group and other even generators which from the Coleman - Mandula theorem commute with the Poincare group, i.e.: they are scalar and pseudoscalar. Hence the most general possibility is of the form:

$$\{Q_a^i, Q_b^j\} = r(\gamma^a C)_{ab} P_a \delta^{ij} + s(\sigma^{ab} C)_{ab} J_{ab} \delta^{ij} + C_{ab} U^{ij} + (\gamma_5 C)_{ab} V^{ij}$$

The fact that numerically invariant tensors have been used, under the Poincare group, is a consequence of the generalised Jacobi identities between two odd and one even generator. (See condition (c) expressed by (II.16))

The even generators $U^{ij} = -U^{ji}$ and $V^{ij} = -V^{ji}$ are called central charges, denoted by Z . It is a consequence of the generalized Jacobi identities that they commute with all other generators including themselves i.e.:

$$[U^{ij}, \text{any generator}] = [V^{ij}, \text{any generator}] = 0$$

The $[P_a, \{Q_a^i, Q_b^j\}] + \dots = 0$ identity implies that $s=0$, and P_a can be normalized by setting $r=2$, yielding the final result :

$$\{Q_a^i, Q_b^j\} = 2(\gamma_a C)_{ab} \delta^{ij} P^a + C_{ab} U^{ij} + (\gamma_5 C)_{ab} V^{ij}$$

Finally the constraints placed on the internal symmetry group by the generalised Jacobi identity, will have to be pursued. Taking (II.27) into consideration, the previous equation gives:

$$\{Q_A^i, \bar{Q}_B^j\} = -2i(\sigma^a)_{AB} \delta_j^i P_a \quad (II.28)$$

$$\{Q_A^i, Q_B^j\} = \epsilon_{AB} (U^{ij} + iV^{ij}) = \epsilon_{AB} Z^{ij} \quad (II.29)$$

$$[Z^{ij}, \text{any generator}] = 0 \quad (II.30)$$

The supersymmetry generators commute with the momenta:

$$[Q_{ai}, P_\mu] = [\bar{Q}^i_a, P_\mu] = 0 \quad (II.31)$$

The proof follows: The commutator of a 'Q' with a 'P_μ' could contain the Lorentz representations (1,1/2)ⁱ and (0,1/2). Since there are no (1,1/2) generators present, the most general possibility is a sum over $\bar{Q}$'s:

$$[Q_{ai}, P_\mu] = c_{ij} (\sigma_\mu)_{a\beta} \bar{Q}^{\beta j}$$

The adjoint of this is:

$$[\bar{Q}^{ai}, P_\mu] = (c_{ij})^* (\bar{\sigma}_\mu)^{a\beta} Q_{\beta j}$$

so that:

$$[[Q_{ai}, P_\mu], P_\nu] = c_{ij} (c_{jk})^* (\sigma_\mu \bar{\sigma}_\nu)_a{}^\beta Q_{\beta k}$$

From $[P_\mu, P_\nu] = 0$, and the Jacobi identity it is to be deduced that for the matrix C with matrix elements c_{ij} : $CC^* = 0$, since the σ -matrix factor does not vanish. The most general $\{Q, Q\}$ allowed - according to (II.29) - is:

$$\{Q_{ai}, Q_{\beta j}\} = 2\epsilon_{a\beta} Z_{[ij]} + \text{a term symmetric in } a\beta$$

The Z^{ij} are a linear combination of internal symmetry generators, and hence commute with P_μ . Then:

$$0 = \epsilon^{a\beta} [\{Q_{ai}, Q_{\beta j}\}, P_\mu] = \epsilon^{a\beta} \{Q_{ai}, [Q_{\beta j}, P_\mu]\} - (i \leftrightarrow j)$$

$$= \epsilon^{a\beta} c_{jk} (\sigma_\mu)_{\beta\beta} \{Q_{ai}, \bar{Q}^{\beta k}\} - (i \leftrightarrow j) \propto c_{[ij]} P_\mu$$

which means that C is symmetric, hence $CC^\dagger = 0$ which implies $c_{ij} = 0$. As a consequence (II.31) is verified.

The Q 's generally also carry some representation of the internal symmetry:

$$[Q_{ai}, B_r] = (b_r)_i^j Q_{aj} \quad (\text{II.32a})$$

Since the internal symmetry group is compact, the representation matrices can be chosen Hermitian: $b_r = b_r^\dagger$. For ' $\bar{Q}$ ' it then holds:

$$[\bar{Q}_a^i, B_r] = -\bar{Q}_a^j (b_r)_j^i \quad (\text{II.32b})$$

The largest possible internal symmetry group which can act non trivially on Q is thus $U(N)$.

The double index structure on the Q 's reflects the direct product structure of the bosonic symmetry group: α and β , belong to the Poincare group and i is the internal symmetry index.

The following equations summarize the results of the previous analysis:

$$[P_\mu, P_\nu] = 0 \quad (\text{II.33a})$$

$$[P_\mu, M_{\rho\sigma}] = i(n_{\mu\rho} P_\sigma - n_{\mu\sigma} P_\rho) \quad (\text{II.33b})$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(n_{\nu\rho} M_{\mu\sigma} - n_{\nu\sigma} M_{\mu\rho} - n_{\mu\rho} M_{\nu\sigma} + n_{\mu\sigma} M_{\nu\rho}) \quad (\text{II.33c})$$

$$[B_r, B_s] = ic_{rs} B_t \quad (\text{II.33d})$$

$$[B_r, P_\mu] = [B_r, M_{\mu\nu}] = 0 \quad (\text{II.33e})$$

$$[Q_{ai}, P_\mu] = [\bar{Q}_a^i, P_\mu] = 0 \quad (\text{II.33f})$$

$$[Q_{ai}, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\nu})_a^\beta Q_{\beta i} \quad (\text{II.33g})$$

$$[\bar{Q}_a^i, M_{\mu\nu}] = -\frac{1}{2}\bar{Q}_a^j (\bar{\sigma}_{\mu\nu})^\beta_j \quad (\text{II.33h})$$

$$[Q_{ai}, B_r] = (b_r)_i^j Q_{aj} \quad (\text{II.33i})$$

$$[\bar{Q}_a^i, B_r] = -\bar{Q}_a^j (b_r)_i^j \quad (\text{II.33j})$$

$$\{Q_{ai}, \bar{Q}_b^j\} = 2\delta_i^j (\sigma^\mu)_{a\beta} P_\mu \quad (\text{II.33k})$$

$$\{Q_{ai}, Q_{bj}\} = 2\epsilon_{a\beta} Z_{ij}, \text{ with } Z_{ij} = a^r_{ij} B_r \quad (\text{II.33l})$$

$$\{\bar{Q}_a^i, \bar{Q}_b^j\} = -2\epsilon_{a\beta} Z^{ij} \text{ with } Z^{ij} = (Z_{ij})^\dagger \quad (\text{II.33m})$$

$$[Z_{ij}, \text{any generator}] = 0 \quad (\text{II.33n})$$

The original - rather loose - definition already given to supersymmetries was based on the concept of a symmetry that involves different spins. However the entire analysis so far

has revealed that supersymmetries can appear in relativistic field theories only in a very specific form: they are generated by spinorial charges which obey well defined anticommutational relations. The present analysis shall henceforth abide by this definition in referring to 'Supersymmetries' or - in the context of curved spacetime - to 'Supergravity'. As was already mentioned, O'Raifeartaigh's theorem imposes the condition that all members of an irreducible multiplet of an internal symmetry group have the same mass. That was due to the fact that: $[B_r, P^2]=0$. It has also been proved that: $[Q, P_\mu]=0$ and consequently $[Q, P^2]=0$. As a result O'Raifeartaigh's theorem still holds for the spinorial charges. As an example of how this theorem results in the aforementioned mass degeneracy, consideration will be made of two supermultiplet states, which for simplicity's sake are expressed in their rest frame: $|A_r\rangle$, $|B_r\rangle$. Then:

$$[Q, P^2]=0 \quad (a)$$

$$\langle A_r | [Q, P^2] | B_r \rangle = \langle A_r | Q P^2 - P^2 Q | B_r \rangle$$

$$= \langle A_r | (m_A^2 - m_B^2) Q | B_r \rangle = (m_A^2 - m_B^2) \langle A_r | Q | B_r \rangle \quad (b)$$

Now since: $\langle A_r | Q | B_r \rangle \neq 0$ in combination with (a) suggests that:

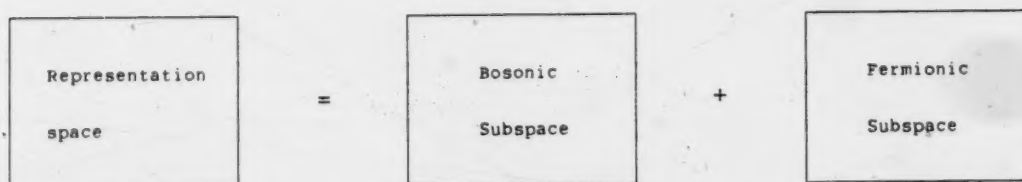
$$m_A = m_B$$

In effect, this leads to the very important conclusion that:

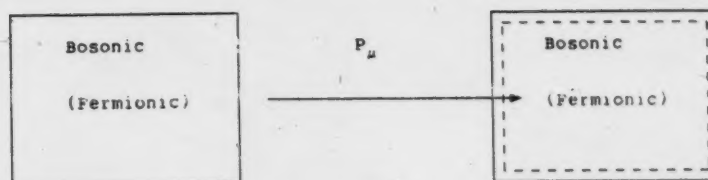
Supersymmetry multiplets contain different spins but are always degenerate in mass and therefore supersymmetry must be broken in nature where elementary particles do not come in mass degenerate multiplets.

To conclude the analysis of the supersymmetry algebra, a discussion of its representations, will be furnished.

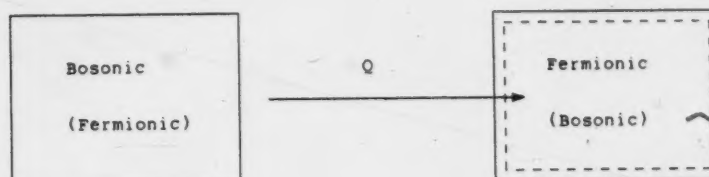
Consistency with the nature of the supersymmetry algebra, requires a grading of the vector space on which it acts. This is equivalent to stating that this representation space can be divided into a bosonic and a fermionic sector in view of the fact that supersymmetry transformations relate fermions with bosons.



The even elements of the supersymmetry algebra (the bosonic generators P_μ , $M_{\mu\nu}$, B_μ) map the subspaces into themselves, and a-priori one is to allow the mapping to be onto a proper subspace of the original subspace:



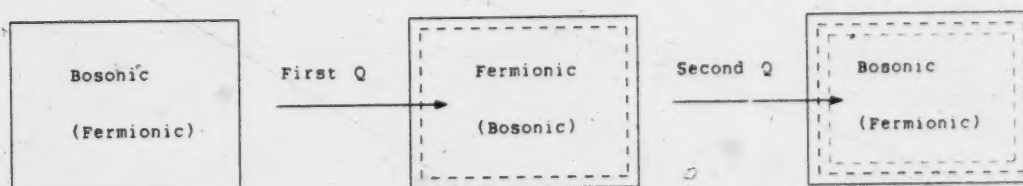
The odd elements (the supersymmetry generators) map the bosonic subspace into the fermionic one and vice - versa:



The anticommutators are a particular linear combination of two subsequent mappings. Especially the relation:

$$\{Q, \bar{Q}\} = 2\sigma^\mu P_\mu$$

characteristic of supersymmetry, implies that there should be a particular way of transiting from one subspace to another and back, such that the net result is an operation with P on the original subspace.



For a large class of representations, the representation $r(P_\mu)$ of the translation operator will map the

representation space onto itself. On a physical multiparticle state, for instance, it holds:

$$r(P_\mu) = (E, \bar{p}) \quad (\text{II.34})$$

with E and $\bar{p}$ being the total momentum of the state. On quantum fields $\phi(x)$, P_μ is the generator of translations: $\phi(x) \rightarrow \phi(x+y)$, and is represented by the partial derivatives:

$$r(P_\mu) = i\partial_\mu \quad (\text{II.35})$$

In both cases no degrees of freedom are lost from the representation space through the action of P . This implies that the dimension of the fermionic subspace must be the same as that of the bosonic subspace. This is so because:

(a) no picture can have more dimensions than its original, and

(b) in the double mapping $Q\bar{Q} + \bar{Q}Q$ no dimensions become lost.

Thus, this discussion leads to the following theorem which holds for the wide class of representations characterized by (II.34) and (II.35):

The fermionic and bosonic subspaces of the representation space of a supersymmetry algebra have the same dimension.

In what follows, this theorem will be occasionally demonstrated in both a supersymmetric field theoretical

model and phenomenology.

III. A SUPERSYMMETRIC ACTION: THE WESS - ZUMINO MODEL

This section will exemplify all previous considerations, through a construction of a specific supersymmetric model which historically was the first renormalizable one in Supersymmetry. The Wess - Zumino model [WES.a74].

The starting point, shall be the following Lagrangian which combines both fermion and boson degrees of freedom:

$$L_0^{WZ} = \frac{1}{2} \partial_\mu S \partial^\mu S + \frac{1}{2} \partial_\mu P \partial^\mu P + \frac{1}{4} \bar{\chi} \gamma^\mu \partial_\mu \chi \quad (\text{III.1})$$

where the kinetic terms for two - in principle scalar - real fields S and P, and the corresponding kinetic term for a Majorana spinor field, have been taken into consideration. Both kinetic terms have a phase invariance of their own and the Lagrangian has two independent global invariances:

$$\chi \rightarrow (e^{i\alpha\gamma_5})\chi, \quad (S+iP) \rightarrow e^{i\beta}(S+iP) \quad (\text{III.2})$$

Any further invariance will involve transformations that change the spinless fields S and P into the spinor field χ . The general characteristics of this transformation are:

1) the transformation parameter is essentially a Grassman spinor field, call it α , a global infinitesimal Majorana parameter.

2) In its simplest form, the transformation of S and P must involve no derivative operator and that of χ must involve one since the fermion kinetic term has one less derivative than the scalar kinetic term. These

considerations, would lead to:

$$\delta(S \text{ or } P) = \bar{\alpha} M \chi \quad (\text{III.3})$$

where M is some 4x4 matrix. Since no vector indices are involved, it must only contain 1 or γ_5 . Hence, it is set to be:

$$\delta S = a \bar{\alpha} \chi \quad (\text{III.4})$$

$$\delta P = i b \bar{\alpha} \gamma_5 \chi \quad (\text{III.5})$$

where a and b are unknown real coefficients. Here, use has been made of the phase invariance (III.2) to define the variation of S to be along 1 and that of P along $i\gamma_5$. The right hand side of the variation is arranged to be real. Then, since ∂_μ is invariant under a supersymmetry transformation, this yields:

$$\delta \left[\frac{1}{2} \partial_\mu S \partial^\mu S + \frac{1}{2} \partial_\mu P \partial^\mu P \right] = (a \partial^\mu S \bar{\alpha} + i b \partial^\mu P \bar{\alpha} \gamma_5) \partial_\mu \chi \quad (\text{III.6})$$

Obviously now, (III.4) and (III.5) imply that on parity conservation grounds, S is a scalar and P a pseudoscalar field.

Having accomplished that, the obvious next step is to determine the variation of χ . The important point to this effect, is that:

$$\delta \left[\frac{1}{4} \bar{\chi} \gamma^{\mu\nu} \partial_\mu \chi \right] = \frac{1}{2} \delta \bar{\chi} \gamma^\mu \partial_\mu \chi - \frac{1}{2} \partial_\mu \bar{\chi} \gamma^\mu \delta \chi \quad (\text{III.7})$$

up to a total divergence. Use will now have to be made of the vector part of the "Majorana - flip" properties:

ξ_n , $\xi_{\gamma_5 n}$, and $\xi_{\gamma_\mu \gamma_5 n}$ are even as $\xi \rightarrow n$, while

$\xi_{\gamma_\mu n}$ and $\xi_{\sigma_{\mu\nu} n}$ are odd. These hold for any Majorana spinors ξ and n . Their application to (III.7) yields:

$$\delta[\frac{1}{4}\bar{\chi}\gamma^\mu\bar{\sigma}^\mu\chi]=\delta\bar{\chi}\gamma^\mu\partial_\mu\chi \quad (\text{III.8})$$

up to surface terms. Then, combination of them all shows that:

$$\delta L_0^{WZ}=(\delta\bar{\chi}\gamma_\mu+a\partial_\mu\bar{S}\bar{\alpha}+ib\partial_\mu\bar{P}\bar{\alpha}\gamma_5)\partial^\mu\chi+ \text{S. T.} \quad (\text{III.9})$$

or, after performing a partial integration:

$$\delta L_0^{WZ}=-(\partial_\mu\delta\bar{\chi}\gamma^\mu+a\partial_\mu\partial^\mu\bar{S}\bar{\alpha}+ib\partial_\mu\partial^\mu\bar{P}\bar{\alpha}\gamma_5)\chi+ \text{S. T.} \quad (\text{III.10})$$

Thus L_0^{WZ} changes only by a total divergence if $\delta\bar{\chi}$ obeys the following equation:

$$\partial_\mu\delta\bar{\chi}\gamma^\mu+a\Box\bar{S}\bar{\alpha}+ib\Box\bar{P}\bar{\alpha}\gamma_5=0 \quad (\text{III.11})$$

A solution is easily found to be:

$$\delta\chi=a\gamma_\rho\alpha\partial^\rho\bar{S}-ib\gamma_\rho\gamma_5\alpha\partial^\rho\bar{P} \quad (\text{III.12})$$

(since: $\gamma_\rho\gamma_\sigma\partial^\rho\partial^\sigma=\partial_\rho\partial^\rho$)

The goal has therefore been achieved. A set of

transformations between spinless and spin 1/2 fields, has been derived, capable of leaving the sum of their kinetic terms invariant (up to a canonical transformation). Consistency requires that these transformations close amongst themselves, or equivalently, that they form a group.

The relevant proof, commences with an examination of the effect of two supersymmetry transformations on the fields. Explicitly:

$$[\delta_1, \delta_2]S = a\bar{\alpha}_2 \delta_1 \chi - (1 \leftrightarrow 2)$$

$$= a\bar{\alpha}_2 [a\gamma_\rho \alpha_1 \partial^\rho S - ib\gamma_\rho \gamma_5 \alpha \partial^\rho P] - (1 \leftrightarrow 2) = 2a^2 \bar{\alpha}_2 \gamma_\rho \alpha_1 \partial^\rho S \quad (\text{III.13})$$

To obtain the last equation the Majorana flip property of the axial vector part, was invoked. At the same time:

$$\delta x^\mu = \epsilon^\rho \partial_\rho x^\mu = \epsilon^\mu \quad (\text{III.14})$$

with $x \equiv (t, \bar{x})$. Then letting:

$$2a^2 \bar{\alpha}_2 \gamma_\rho \alpha_1 = \epsilon_\rho \quad (\text{III.15})$$

results in:

$$[\delta_1, \delta_2]S = \epsilon_\rho \partial^\rho S \quad (\text{III.16})$$

Obviously, this is tantamount to stating that the effect of

two supersymmetry transformations on the field S is to translate it by an amount $2ia^2\bar{\alpha}_2\gamma_\rho\alpha_1$, as was expected on the basis of the supersymmetry algebra. As regards to the field P , it holds:

$$[\delta_1, \delta_2]P = ib\bar{\alpha}_2\gamma_5\delta_1\chi - (1+2) = 2b^2\bar{\alpha}_2\gamma_\rho\alpha_1\partial^\rho P \quad (\text{III.17})$$

where again the Majorana flip identity for axial vector has been used. Since transformations must be the same for S , P and χ it should hold:

$$b = \pm a \quad (\text{III.18})$$

On the same lines, a verification of the relevant translational effect of two successive supersymmetry transformations on the field χ , will have to be furnished:

$$\begin{aligned} [\delta_1, \delta_2]\chi &= a\gamma_\rho\alpha_2\partial^\rho\delta_1 S - ib\gamma_\rho\gamma_5\alpha_2\partial^\rho\delta_1 P - (1+2) \\ &= a^2\gamma_\rho\alpha_2\bar{\alpha}_1\partial^\rho\chi + b^2\gamma_\rho\gamma_5\alpha_2\bar{\alpha}_1\gamma_5\partial^\rho\chi - (1+2) \end{aligned} \quad (\text{III.19})$$

It would be desirable to rewrite the right hand side of this equation in a form similar to the others, that is involving $\bar{\alpha}_2\gamma_\rho\alpha_1$ and not the matrix $\bar{\alpha}_2\alpha_1$, that appears in (III.19). For this the so called "Fierz rearrangement" is invoked [RAM.81].

Application to the case in question, yields:

$$\alpha_2 \bar{\alpha}_1 - \alpha_1 \bar{\alpha}_2 = -\frac{1}{2} \bar{\alpha}_1 \gamma^\rho \alpha_2 \gamma_\rho + \bar{\alpha}_1 \sigma^{\rho\sigma} \alpha_2 \sigma_{\rho\sigma} \quad (\text{III.20})$$

where again use has been made of the Majorana flip properties.

Combining (III.19), with (III.18) and (III.20) yields:

$$[\delta_1, \delta_2] \chi = a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \gamma_\mu \partial^\rho \chi \quad (\text{III.21})$$

use of the anticommutator of the matrices, results in:

$$[\delta_1, \delta_2] = 2a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \partial_\mu \chi - a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \gamma_\mu \gamma^\rho \partial_\rho \chi \quad (\text{III.22})$$

The first term on the right hand side is the expected result, but unfortunately, there is an additional term proportional to $\gamma^\rho \partial_\rho \chi$. This extra term vanishes whenever the classical equations of motion are valid, (on shell representation). Elimination of this term would seem to require an enlargement of the definition of $\delta \chi$. The interesting observation, at this point, is that an addition to $\delta \chi$ of (III.12) of an extra variation of the form:

$$\delta_{\text{extra}} \chi = (F + i\gamma_5 G) \alpha \quad (\text{III.23})$$

where F and G are functions of x, the relations (III.16) and (III.17) are not affected because of the Majorana flip conditions. For example:

$$[\delta_1, \delta_2]_{\text{extra}} S = a \bar{\alpha}_2 \delta_{1-\text{extra}} \chi - (1+2) = a \bar{\alpha}_2 (F + i\gamma_5 G) \alpha_1 - (1+2) = 0 \quad (\text{III.24})$$

However this extra variation brings about a contribution on χ , namely:

$$[\delta_1, \delta_2]_{\text{extra}} \chi = (\delta_1 F + i \gamma_5 \delta_1 G) \alpha_2^{-1+2} \quad (\text{III.25})$$

The extra term in (III.22) can be rewritten in a suggestive form by means of the Fierz rearrangement:

$$\bar{\alpha}_2 \gamma^\mu \alpha_1 \gamma_\mu = \alpha_1 \bar{\alpha}_2 - \gamma_5 \alpha_1 \bar{\alpha}_2 \gamma_5^{-1+2} \quad (\text{III.26})$$

Now, comparison with (III.25) shows that the choice:

$$\delta_1 F = a^2 \bar{\alpha}_1 \gamma^\rho \partial_\rho \chi \quad (\text{III.27})$$

$$\delta_1 G = -i a^2 \bar{\alpha}_1 \gamma_5 \gamma^\rho \partial_\rho \chi \quad (\text{III.28})$$

cancels the extra term and obtain the desired result. One can easily show that the operator relation:

$$[\delta_1, \delta_2] = 2a^2 \bar{\alpha}_2 \gamma^\mu \alpha_1 \partial_\mu \quad (\text{III.29})$$

is satisfied when acting on F and G . However the new $\delta \chi$, does not leave the original Action invariant because of $\delta_{\text{extra}} \chi$. Nevertheless, it holds that:

$$\begin{aligned} \delta_{\text{extra}} L_0^{\text{WZ}} &= \delta_{\text{extra}} \bar{\chi} \gamma^\mu \partial_\mu \chi = i \bar{\alpha} \gamma^\rho \partial_\rho \chi F - \bar{\alpha} \gamma_5 \gamma^\rho \partial_\rho \chi G \\ &= -\frac{1}{2a^2} \delta[F^2 + G^2] \end{aligned} \quad (\text{III.30})$$

Therefore the Action:

$$S_0^{WZ} = \int d^4x \left[\frac{1}{2} \partial_\mu S \partial^\mu S + \frac{1}{2} \partial_\mu P \partial^\mu P + \frac{1}{4} \bar{\chi} \gamma^{\rho\sigma} \partial_\rho \chi + \frac{1}{2a^2} (F^2 + G^2) \right] \quad (\text{III.31})'$$

is invariant under the supersymmetry transformations:

$$\delta S = a \bar{\alpha} \chi \quad (\text{III.32a})$$

$$\delta P = i a \bar{\alpha} \gamma_5 \chi \quad (\text{III.32b})$$

$$\delta F = a^2 \bar{\alpha} \gamma^\rho \partial_\rho \chi \quad (\text{III.32c})$$

$$\delta G = -i a^2 \bar{\alpha} \gamma_5 \gamma^\rho \partial_\rho \chi \quad (\text{III.32d})$$

$$\delta \chi = a \gamma_\rho \alpha \partial^\rho S - i a \gamma_\rho \gamma_5 \alpha \partial^\rho P + (F + i \gamma_5 G) \alpha \quad (\text{III.32e})$$

These transformations now all satisfy the operator equation (III.29).

With the introduction of the auxiliary fields F and G , there is now an equal number of spinless (S , P , F and G) and spinor (the four real components of χ) fields irrespective of the equations of motion. Obviously on shell where the classical equations of motion hold, the two auxiliary fields can be eliminated on account of their lack of kinetic terms, whereas at the same time the Majorana spinor bears only two degrees of freedom the other two having been eliminated by the Dirac equation. Consequently this supersymmetric model clearly exemplifies the theorem proved in the previous chapter according to which the fermionic and bosonic

subspaces of the representation space of a supersymmetry algebra, have the same dimension. Such auxiliary fields are necessary in all supersymmetric models with more than one fermionic generators (extended supersymmetries); their description in these cases has proved one of the outstanding difficulties in the development of the subject [FAY.77].

Now, consideration of a space-time dependent α (however, without any derivatives of α in (III.32)), shows that the variation of L_0^{WZ} can only be proportional to $\partial_\mu \alpha$:

$$\delta L_0^{WZ} = (\partial^\mu \bar{\alpha}^\nu) J_{\mu\nu} \quad (\text{III.33})$$

where $J_{\mu\nu}$ is the supercurrent [WES.b74]:

$$J_{\mu\nu} = [\gamma^\lambda \partial_\lambda (S - \gamma_5 P)] \gamma_\mu \chi_\nu \quad (\text{III.34})$$

The supercurrent is the usual Noether current of the symmetry in consideration, that is supersymmetry, as is to be expected by analogy to the Noether current associated with translational invariance (being the energy - momentum tensor $T_{\mu\nu}$) and that associated with Lorentz invariance (being the rotation tensor $J_{\mu\nu\lambda}$). Since the generator of SUSY, Q_a , now has a spinor index it is to be expected that the associated Noether current is a vector-spinor, as it is from (III.34). Finally, it is worth noticing that:

$$\partial^\mu J_{\mu a} = 0 \quad (\text{III.35})$$

if the free field equations of motion $\square S = \square P = \not{\partial} \chi = 0$, are used,

as again expected for a Noether current of a global symmetry. The supercharge $Q_a = \int d^3x J_{0a}$ is therefore a constant in time. Furthermore from (III.29), it becomes obvious that this model entails yet another theorem, which was emphasized - in the previous chapter - as being of central importance to supersymmetry: That the effect of two successive supersymmetry transformations, is equivalent to a translation in space-time. Possible mass and interaction terms may now be added to (III.31), being invariant under (III.32). The mass term can be seen, on dimensional and parity conserving grounds [in the light of (III.32)], as involving $\bar{\chi}\chi$, SF and PG . Its detailed form, in order to have invariance under (III.32), is easily seen to be:

$$L_m^{WZ} = -i\frac{1}{2}m(\bar{\chi}\chi - a^{-1}FS - a^{-1}PG) \quad (III.36)$$

Elimination of the auxiliary fields F and G from $(L_0^{WZ} + L_m^{WZ})$ by means of their equations of motion:

$$F = -mS, \quad G = -mP \quad (III.37)$$

gives the usual mass terms: $-\frac{1}{2}m^2(S^2 + P^2)$. These terms and the spinor term in eq.(III.36) could have been considered without any auxiliary fields, but that would have the disadvantage off-shell of non closure of the algebra.

A similar situation arises for the interaction terms. If only a dimensionless coupling constant is allowed on grounds of renormalizability, then invariant parity-conserving cubic

terms of dimension four, can be constructed in a way that they are at most linear in the auxiliary fields F and G [so they can still be eliminated algebraically through (III.37)] as proportional to FS^2 , FP^2 , GSP , $\bar{X}XS$, $\bar{X}\gamma_5XP$. This supersymmetric Yukawa coupling term which leaves one global chiral invariance can be either:

$$L_{int.}^{WZ} = ig[\bar{X}XS - i\bar{X}\gamma_5XP + F(P^2 - S^2) - 2GSP] \quad (III.38)$$

or

$$L_{int.}^{WZ} = ig'[\bar{X}XP + i\bar{X}\gamma_5XS + G(S^2 - P^2) - 2FSP] \quad (III.39)$$

The way invoked to introduce the mass and interaction terms in the supersymmetric lagrangian will be generalized and become much more transparent by taking advantage of the elegant "superspace formalism" which is to be introduced shortly, in the next chapter.

At the moment it suffices to say that other cubic or quartic or higher order terms, could have also been added to $L_{int.}^{WZ}$ of (III.38) or (III.39). If these involved the auxiliary fields F and G non trivially, they would necessarily have required the introduction of couplings with a positive length dimension, or equivalently negative mass dimension. However - it is well established in Quantum Field Theory - that coupling constants with negative mass dimensions (for $\hbar=c=1$), lead to non-renormalizable theories. The only other terms allowed with a dimensionless coupling

constant would be proportional to a fourth order polynomial in S and P . This can easily be seen to be non-invariant under (III.32) for any choice of coefficients in that polynomial, so it can be discarded. Apparently then, the lagrangian for the Wess - Zumino model is the sum of the three terms expressed by the lagrangian which corresponds to the action (III.31) the mass term given by (III.36), and the interaction lagrangian given by either (III.38) or (III.39).

As will be demonstrated in due time, the quantum field theory of the model described by $(L_0^{WZ} + L_m^{WZ} + L_{int}^{WZ})$, can be quantized in a manner preserving supersymmetry, and can be shown to be less ultra-violet divergent than one would have expected on simple power-counting arguments. Only a single logarithmic divergence is present, leading to a common wave function renormalization of the fields S , P and χ and with mass and charge renormalization also being determined by this wave function renormalization constant Z as:

$$m_r = Zm, \quad g_r = Z^{3/2}g \quad (III.40)$$

The particularity of this situation, will also be discussed in due time.

However, a rigorous discussion of Quantum Field Theoretical techniques on supersymmetric models, necessitates the introduction of the concept of superspace and the corresponding formalism developed on it, which is to be pursued henceforth.

IV. SUPERSPACE AND SUPERFIELDS

The task ahead is to obtain field theoretical realizations of supersymmetry, which will in effect necessitate representations in terms of local fields. Such representations were first obtained by trial and error, before the elegant method which makes use of the concept of superspace was put forward [SAL.75].

The construction of a supersymmetric field theory is greatly simplified by the introduction of anticommuting parameters, θ^a , θ_a :

$$\{\theta^a, \theta^b\} = \{\theta^a, Q_\beta\} = \dots = \{P_m, \theta^a\} = 0 \quad (\text{IV.1})$$

and similar equations for θ_a .

Using these anticommuting parameters, the supersymmetry algebra may be written in terms of commutators in the more familiar form of a Lie Algebra [ROS.85]:

$$[\theta Q, \bar{\theta} Q] = 2\theta \sigma^\mu \bar{\theta} P_\mu \quad (\text{IV.2})$$

$$[\theta Q, \theta Q] = [\bar{\theta} Q, \bar{\theta} Q] = 0 \quad (\text{IV.3})$$

$$[P^\mu, \theta Q] = [P^\mu, \bar{\theta} Q] = 0 \quad (\text{IV.4})$$

where use has been made of the notation:

$$\theta Q = \theta^a Q_a, \quad \bar{\theta} Q = \bar{\theta}_a \bar{Q}^a \quad (\text{IV.5})$$

with θ and $\bar{\theta}$ being Weyl spinors. The task is to find a

representation of the supersymmetry algebra, in terms of differential operators. The natural parameter space for these operators has eight dimensions, four (the usual Minkowski space) associated with the operator P , and four with the spinors Q and $\bar{Q}$. The last four coordinates however, are not numbers but elements of a Grassman algebra (see appendix). An element of this space will be denoted by $z^\mu = \{x^\mu, \theta^a, \bar{\theta}^a\}$. This eight dimensional space is called "superspace".

The algebra expressed by (IV.2), (IV.3) and (IV.4), can now formally be integrated in order to obtain a group.

A "finite" group element can be defined by:

$$G(x, \theta, \bar{\theta}) = \exp[i(\theta^a Q_a + \bar{\theta}^a \bar{Q}_a - x_\mu P^\mu)] \quad (\text{IV.6})$$

It is easy to compute the product of two group elements using Housdorf's formula:

$$e^A e^B = e^{A+B + \frac{1}{2}[A, B] + \dots} \quad (\text{IV.7})$$

To get:

$$G(y, \xi, \bar{\xi}) G(x, \theta, \bar{\theta}) =$$

$$G(y_\mu + x_\mu - i\xi^a \sigma_{\mu aa} \bar{\theta}^a + i\theta^a \sigma_{\mu aa} \bar{\xi}^a, \xi^a + \theta^a, \bar{\xi}^a + \bar{\theta}^a) \quad (\text{IV.8})$$

This implies that the group element induces a motion of the parameter space into itself:

$$G(y, \xi, \bar{\xi}) : \{x, \theta, \bar{\theta}\} \rightarrow \{x+y-i\xi\sigma\bar{\theta}+i\theta\sigma\bar{\xi}, \xi+\theta, \bar{\xi}+\bar{\theta}\} \quad (\text{IV.9})$$

Equation (IV.9), shows that supersymmetry transformations can act on superspace as generalized translations.

A superfield is a function of the superspace element $z^M : \phi(x, \theta, \bar{\theta})$. A superfield may have Lorentz, tensor or spinor indices. A transformation of the group acts on it as a generalized translation:

$$G(y, \xi, \bar{\xi})\phi(x, \theta, \bar{\theta}) = \phi(x+y-i\xi\sigma\bar{\theta}+i\theta\sigma\bar{\xi}, \theta+\xi, \bar{\theta}+\bar{\xi}) \quad (\text{IV.10})$$

The interest in superfields derives from the fact that, like any function of Grassman variables, they are polynomials in θ and $\bar{\theta}$:

$$\begin{aligned} F(x, \theta, \bar{\theta}) = & f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \bar{\theta}\bar{\theta} n(x) \\ & + \theta\sigma^m\bar{\theta}v_m(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta y(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \end{aligned} \quad (\text{IV.11})$$

All higher powers of θ , $\bar{\theta}$ vanish because of the anticommutation relations. An infinitesimal supersymmetry transformation yields:

$$\delta F(x, \theta, \bar{\theta}) = (\xi Q + \bar{\xi} \bar{Q}) F(x, \theta, \bar{\theta}) \quad (\text{IV.12})$$

A scalar superfield is defined to transform under supersymmetry as:

$$\phi'(x', \theta') = \phi(x, \theta) \quad (\text{IV.13})$$

At the same time it is known that superspace is an eight dimensional manifold, parameterized by the coordinates x and θ which obey the relations [WEST.86]:

$$[x_\mu, x_\nu] = 0 \quad (IV.14a)$$

$$[x_\mu, \theta_a] = 0 \quad (IV.14b)$$

$$\{\theta_a, \theta_\beta\} = 0 \quad (IV.14c)$$

As will be shown, in due time, the supersymmetry transformations and translations are realized on this manifold by:

$$x'_\mu = x_\mu - i\bar{\theta}\sigma_\mu\theta + a_\mu \quad (IV.15)$$

$$\theta'_a = \theta_a + \epsilon_a \quad (IV.16)$$

where since the analysis makes use of the Weyl representation, ϵ_a is always an anticommuting Weyl spinor. Consequently an infinitesimal supersymmetric transformation on the field ϕ , results in:

$$\delta\phi = \phi'(x', \theta') - \phi(x, \theta) = \phi(x_\mu - i\bar{\theta}\sigma_\mu\bar{\theta}, \theta + \epsilon) - \phi(x, \theta) = \theta Q\phi \quad (IV.17)$$

At the same time:

$$\delta\phi(x, \theta) = \frac{\partial\phi(x, \theta)}{\partial x} dx + \frac{\partial\phi(x, \theta)}{\partial\theta_a} \theta_a \quad (\text{IV.18})$$

with dx given by (IV.15) as:

$$dx_\mu = i\bar{\theta}\sigma_\mu\theta \quad (\text{IV.19})$$

(allowing for the translational constant a). Equating the left hand side of (IV.17) with the corresponding one of (IV.18), and using (IV.19), renders the required representation of the graded Lie algebra expressed by (IV.2), (IV.3), (IV.4) in terms of differential operators:

$$Q_a = \frac{\partial}{\partial\theta^a} - i\sigma^m_{aa} \bar{\theta}^a \partial_m \quad (\text{IV.20})$$

$$\bar{Q}^a = \frac{\partial}{\partial\bar{\theta}_a} - i\theta^a \sigma^m_{a\beta} \epsilon^{\beta a} \partial_m \quad (\text{IV.21})$$

$$P_\mu = i \frac{\partial}{\partial x^\mu} \quad (\text{IV.22})$$

Using (IV.15), it would be convenient to re-express Q and $\bar{Q}$, in terms of the variables y , θ and $\bar{\theta}$:

$$Q_a = \frac{\partial}{\partial\theta^a} \quad (\text{IV.23})$$

$$\bar{Q}_a = -\frac{\partial}{\partial\bar{\theta}^a} + 2i\theta^a \sigma^m_{aa} \frac{\partial}{\partial y^m} \quad (\text{IV.24})$$

Now using the representation for Q and $\bar{Q}$, together with the absence of powers of θ and $\bar{\theta}$ higher than two, shows immediately that δF is also in the form of a general

superfield and so superfields form linear representations of the supersymmetry algebra. These representations are generally reducible, necessitating - as a result - constraints which would give rise to the superfields corresponding to a given irreducible representation.

In order to construct actions, it is desirable to first find covariant derivatives. Although the operator $\partial_m = P_m$, commutes with supersymmetry as $\{Q_a, P_m\} = 0$, the spinorial derivative $\frac{\partial}{\partial \theta^a}$ does not transform covariantly:

$$\frac{\partial}{\partial \theta^a} = \frac{\partial \bar{\theta}^{\beta}}{\partial \theta^a} \frac{\partial}{\partial \bar{\theta}^{\beta}} + \frac{\partial x'^{\nu}}{\partial \theta^a} \frac{\partial}{\partial x'^{\nu}} = \frac{\partial}{\partial \bar{\theta}^a} + i \sigma^m_{ab} \theta^b \frac{\partial}{\partial x'^m}$$

The derivative given by :

$$D_a = \frac{\partial}{\partial \theta^a} + i \sigma^m_{ab} \theta^b \frac{\partial}{\partial x'^m} = \frac{\partial}{\partial \theta^a} + 2i \sigma^m_{ab} \bar{\theta}^b \frac{\partial}{\partial y^m} \quad (\text{IV.25})$$

however, does transform covariantly. This can also be checked by confirming that its commutator with the supercharge Q , vanishes:

$$\{Q_a, D_b\} = 0 \quad (\text{IV.26})$$

Obviously then the expression:

$$D_a = -\frac{\partial}{\partial \theta^a} - i \theta^a \sigma^m_{aa} \frac{\partial}{\partial x'^m} = -\frac{\partial}{\partial \theta^a} \quad (\text{IV.27})$$

is a covariant derivative as well.

The transformation properties of what is going to be defined as a chiral supermultiplet, will now be constructed. In that, the complex scalar field is denoted by A and the spinor field is denoted by ψ . As is already known a fermion of definite helicity has two degrees of freedom, corresponding to the two spinor indices in the Weyl basis. In a chiral supermultiplet, supersymmetry relates this state to a scalar which must be complex in order to have the same number of degrees of freedom. Then under an infinitesimal supersymmetry transformation, parameterized by the anticommuting parameters ξ , A and ψ transform, plausibly as:

$$\delta_{\xi} A = (\xi Q + \bar{\xi} \bar{Q}) A \quad (\text{IV.28})$$

$$\delta_{\xi} \psi = (\xi Q + \bar{\xi} \bar{Q}) \psi \quad (\text{IV.29})$$

where the operation of Q and $\bar{Q}$ on A and ψ , must be determined.

By definition then, ψ is the spinor into which A transforms under the action of Q . Then, since $\bar{Q}$ annihilates A , the result is:

$$\delta_{\xi} A = \sqrt{2} \xi \psi \quad (\text{IV.30})$$

with $\sqrt{2}$ being a convenient normalization factor. It is worth noting that the supersymmetry algebra requires that Q have mass dimensions $1/2$, since the anticommutator of two Q 's is proportional to

P_m , with mass dimension 1. This is reflected in (IV.30), a scalar field (dimension 1), is transformed by Q into a spinor field (mass dimension 3/2).

The field ψ , must transform under Q , into either a tensor field of higher dimension 2 (call it F) or, under $\bar{Q}$ into the derivative of A , these being the only terms consistent with the addition of mass dimension 1/2, from Q or $\bar{Q}$ together with the spinor field mass dimension 3/2. The only possible combinations are:

$$\delta_\xi \psi = i\sqrt{2} \sigma^m \bar{\xi} \partial_m A + \sqrt{2} \xi F \quad (\text{IV.31})$$

In this equation, the coefficient of $\partial_\mu A$ is chosen so that:

$$\delta_n \delta_\xi A = 2i \xi \sigma^m \bar{n} \partial_m A + 2\xi n F \quad (\text{IV.32})$$

and thus:

$$(\delta_n \delta_\xi - \delta_\xi \delta_n) A = -2i (\bar{n} \sigma^m \xi - \bar{\xi} \sigma^m n) \partial_m A = 2 (\bar{n} \sigma^m \xi - \bar{\xi} \sigma^m n) P_m A \quad (\text{IV.33})$$

as is required from the supersymmetry algebra.

Similarly, it is found:

$$\begin{aligned} (\delta_n \delta_\xi - \delta_\xi \delta_n) \psi &= -2i (\bar{n} \sigma^n \xi - \bar{\xi} \sigma^n n) \partial_n \psi \\ &\quad - i \sigma^n \bar{\sigma}^m \partial_m \psi [\bar{n} \sigma^n \xi - \bar{\xi} \sigma^n n] + \sqrt{2} (\xi \delta_n F - n \delta_\xi F) \end{aligned} \quad (\text{IV.34})$$

In order for this to satisfy the supersymmetry algebra, it

is required:

$$\delta_\xi F = i\sqrt{2} \bar{\xi} \sigma^n \partial_n \psi \quad (\text{IV.35})$$

Again here - as was the case in the previous chapter - F is an auxiliary field. If the theory in question is noninteracting then the field ψ satisfies the field equations:

$$-i\not{\partial}\psi = -i\bar{\sigma}^n \partial_n \psi = m\bar{\psi} \quad (\text{IV.36})$$

and in this case the field F can be chosen to satisfy:

$$F = -mA^* \quad (\text{IV.37})$$

However, in the presence of interactions, as was discussed in the previous chapter, the field equations change, and consequently F is bound to change as well. For this reason it is convenient to work with an off-shell chiral supermultiplet. Actually, the chiral supermultiplet that has just been constructed, consisting of the scalar field A , the spinor field ψ and the auxiliary field F , describes the simplest renormalizable supersymmetric model: the Wess - Zumino model, that was analysed out of the context of superspace, in the previous chapter. The only difference is that contrary to the previous analysis there, which involved two real scalar fields and two real auxiliary ones, here a complex scalar field and a complex auxiliary one, were incorporated in the chiral supermultiplet, a fact which

evidently amounts the same, in the sense that, the number of degrees of freedom, for bosons, remains the same as the one for fermions.

The set of fields A , ψ and F form a linear representation of the supersymmetry algebra. They are known as "chiral supermultiplet", because the fermion field has a definite chirality.

Under a supersymmetry transformation parameterized by the anticommuting variable θ , the scalar field component $A(x)$ transforms as:

$$A(x) \rightarrow \exp\{\theta Q + \bar{\theta} \bar{Q}\} A(x) = e^{\theta Q} e^{\bar{\theta} \bar{Q}} e^{i(-i\theta\sigma^m\bar{\theta}P_m)} A(x) \quad (\text{IV.38})$$

or:

$$A(x) \rightarrow e^{\theta Q} e^{\bar{\theta} \bar{Q}} A(y) \quad (\text{IV.39})$$

where:

$$y^m = x^m + i\bar{\theta}\sigma^m\theta \quad (\text{IV.40})$$

In deriving this equation, use was made of Hausdorff's formula again, always recalling that the product of more than two Grassman variables vanishes, by the anticommutation relations of the Grassman algebra, which allows for the retaining of the term up to the single commutator in the corresponding exponential.

By complex conjugating (IV.40) yields:

$$y^{\dagger m} = x^m - i\bar{\theta}\sigma^m\theta \quad (\text{IV.41})$$

Obviously now the most obvious generalization of (IV.41) is (IV.15) which had earlier been introduced in an ad-hoc way. Equation (IV.39), may be evaluated by expanding the exponential up to the second powers in θ and $\bar{\theta}$, noting that $\bar{Q}$ annihilates A , and using the results of (IV.30), and (IV.31). This gives:

$$A(x) + \phi(x) = A(y) + \sqrt{2} \psi(y)\theta + F(y)\theta\theta \quad (\text{IV.42})$$

The quantity $\phi(y)$, is known as a chiral superfield, and as usual under an infinitesimal supersymmetry transformation, it changes by:

$$\delta\phi(y, \theta, \bar{\theta}) = (\xi Q + \bar{\xi}\bar{Q})\phi(y, \theta, \bar{\theta}) \quad (\text{IV.43})$$

Combining this with the representation of supersymmetry generators given by (IV.23) and (IV.24), results in:

$$\begin{aligned} \delta\phi(y, \theta, \bar{\theta}) &= \sqrt{2} \xi\psi(y) + [2\xi F(y) + 2i\xi\sigma^m\partial_m A(y)]\theta + i\sqrt{2} \xi\bar{\sigma}^m\partial_m \psi\theta\theta \\ &= \delta_\xi A(y) + \sqrt{2} \delta_\xi \psi(y)\theta + \delta_\xi F(y)\theta\theta \end{aligned} \quad (\text{IV.44})$$

Equating equal powers of θ , yields:

$$\delta_\xi A(y) = \sqrt{2} \xi\psi(y) \quad (\text{IV.45})$$

$$\delta_{\xi}\psi(y)=\sqrt{2} (\xi F(y)+i\xi\sigma^m\partial_m A(y)) \quad (IV.46)$$

$$\delta_{\xi}F(y)=i\sqrt{2} \xi\sigma^m\partial_m\psi \quad (IV.47)$$

These are just the transformation properties of the component fields in equations (IV.30), (IV.31), (IV.35), simply derived by rotating in superspace the scalar component of the chiral field.

The chiral superfield in (IV.42), was defined by:

$$\phi(y)=e^{i\theta Q}e^{i\bar{\theta}\bar{Q}}A(y)=e^{i\theta Q}A(y), (\bar{Q}A=0) \quad (IV.48)$$

and so obviously satisfies the condition:

$$\bar{D}_a\phi=0 \quad (IV.49)$$

As has already been stressed, the linear representations expressed by superfields in the general form (IV.11), are in principle reducible, and hence constraints will have to be imposed, in order to obtain the superfields corresponding to a given irreducible representation. The superfield corresponding to a given representation of supersymmetry is easily found by construction from the component field of lower dimension, as in the example given above for the chiral superfield. The constraint given by (IV.49) expressing the covariance of the field (covariant constraint), makes the corresponding supermultiplet irreducible under supersymmetric transformations, and it is

clear that the only general superfield satisfying this constraint is that given by (IV.42) which in terms of the spacetime variable x , is given by:

$$\begin{aligned} \phi = & A(x) + i\theta\sigma^m\bar{\theta}\partial_m A(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}A(x) \\ & + \sqrt{2}\theta\psi(x) - i2^{-1/2}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x) \end{aligned} \quad (\text{IV.50})$$

where use was made of the definition of y^m , given by equation (IV.40) in combination with (IV.42).

The superfield $\phi^\dagger$ satisfies the constraint:

$$D_a\phi^\dagger = 0 \quad (\text{IV.51})$$

and may be written:

$$\phi^\dagger = A^*(y^\dagger) + \sqrt{2}\bar{\theta}\bar{\psi}(y^\dagger) + \bar{\theta}\bar{\theta}F^*(y^\dagger) \quad (\text{IV.52})$$

with $y^\dagger$ given by (IV.41). It clearly does not satisfy the constraint (IV.49). Both ϕ and $\phi^\dagger$ are known as chiral superfields but ϕ , contains a left handed fermion whereas $\phi^\dagger$ contains a right handed one, a fact which suggests that the latter be called: a right handed chiral superfield, and the former: a left handed chiral one, which is the standard conventional terminology.

Another constraint which results in an irreducible supermultiplet, is the reality constraint, according to which a superfield V obeys the equation:

$$V=V^\dagger \quad (IV.53)$$

As will become manifest shortly, this is the correct constraint to project a vector superfield, which may equally well be obtained by construction from the component field of lowest dimension. The general superfield satisfying this constraint may be written in the form:

$$\begin{aligned} V(x, \theta, \bar{\theta}) = & C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) + \frac{1}{2}i\theta\theta[M(x) + iN(x)] \\ & - \frac{1}{2}i\bar{\theta}\bar{\theta}[M(x) - iN(x)] - i\theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}[\bar{\chi}(x) + \frac{1}{2}i\bar{\sigma}^m\partial_m\chi(x)] \\ & - i\bar{\theta}\bar{\theta}\theta[\chi(x) + \frac{1}{2}i\sigma^m\partial_m\bar{\chi}(x)] + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}[D(x) + \frac{1}{2}\square C(x)] \end{aligned} \quad (IV.54)$$

where the component fields C , D , M , N and the vector field v_m , are all real. In building a supersymmetric gauge field theory it would be desirable that all vector fields be vector gauge bosons, for this is the only way to build a renormalizable theory. To do this it is imperative that the supersymmetric version of gauge invariance be introduced. Wess and Zumino showed that this requires V transformations under a local $U(1)$ gauge group as:

$$V \rightarrow V + \phi + \phi^\dagger \quad (IV.55)$$

where ϕ is a chiral superfield [WES.82].

At this point it should be emphasized, that use has so far been made, of the concept of the vector superfield as

the supermultiplet which contains a vector field, or equivalently satisfies (IV.53).

From (IV.50) it is inferred that:

$$\begin{aligned}
 \phi + \phi^\dagger &= A + A^* + \sqrt{2} (\theta\psi + \bar{\theta}\bar{\psi}) + \theta\theta F + \bar{\theta}\bar{\theta} F^* \\
 &+ i\theta\sigma^m\bar{\theta}\partial_m(A - A^*) + i2^{-1/2}\theta\theta\bar{\theta}\bar{\sigma}^m\partial_m\psi \\
 &+ i2^{-1/2}\bar{\theta}\bar{\theta}\theta\sigma^m\bar{\theta}\partial_m\bar{\psi} + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\square(A + A^*)
 \end{aligned}
 \tag{IV.56}$$

This combination has the gradient $i\partial_m(A - A^*)$ as coefficient of $\bar{\theta}\sigma^m\theta$, and that was the motivation for the supersymmetric generalization of a gauge transformation given in (IV.55), in the sense that according to it, v_m transforms as $v_m \rightarrow v_m - i\partial_m(A - A^*)$, the correct form for a vector field transforming under an abelian gauge transformation. However (IV.55) is a supersymmetric gauge transformation and the other components of V also change according to:

$$C \rightarrow C + A + A^* \tag{IV.57a}$$

$$\chi \rightarrow \chi - i\sqrt{2}\psi \tag{IV.57b}$$

$$M + iN \rightarrow M + iN - 2iF \tag{IV.57c}$$

$$v_m \rightarrow v_m - i\partial_m(A - A^*) \tag{IV.57d}$$

$$\lambda \rightarrow \lambda$$

(IV.57e)

$$D \rightarrow D$$

(IV.57f)

Attention is to be drawn to the fact that in eq.(IV.54) the choice of the components of $\theta\theta\bar{\theta}$ and $\theta\theta\theta\bar{\theta}$, renders λ and D gauge invariant. The fields C , X , M , N and D , are all auxiliary fields which may be eliminated by the equations of motion. However it is possible to remove most of this complexity by taking advantage of the gauge freedom allowed by (IV.57). As a result a gauge is chosen - in the present case corresponding to a specific choice of the chiral superfield - which sets C , X , M , and N to zero. In this gauge - frequently referred to as the Wess Zumino gauge - the vector superfield assumes the form:

$$V = -\bar{\theta}\sigma^m\theta v_m(x) + i\theta\theta\bar{\theta}\lambda(x) - i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\bar{\theta}\bar{\theta}D(x) \quad (\text{IV.58})$$

Although fixing the gauge in this way breaks the manifest supersymmetry, since now $A+A^*$ and $A-A^*$ are independent functions, one still has the usual gauge transformations, $v_m \rightarrow v_m - i\partial_m(A-A^*)$ without introducing the terms eliminated by the supersymmetric gauge fixing. As (IV.58) shows, the vector superfield V contains the vector field v_m , along with a (left handed) fermion, λ . D is an auxiliary field which does not propagate and as usual will be eliminated by the equations of motion.

The obvious next step, involves an attempt towards the generalization of the "U(1) gauge transformation", to a non abelian one. This necessitates the introduction of a chiral superfield ϕ transforming under the non Abelian group G as:

$$\phi \rightarrow e^{-ig\Lambda} \phi \quad (\text{IV.59})$$

where Λ is the matrix:

$$\Lambda_{ij} = F_{ij}^a \Lambda_a \quad (\text{IV.60})$$

The field Λ_a , must be a chiral superfield if the right hand side of (IV.59), is to transform as a chiral superfield, since the condition $\bar{D}_a \phi' = \bar{D}_a e^{-ig\Lambda} \phi = 0$, clearly requires that: $\bar{D}_a \Lambda = 0$. The matrices F_a are the generators of the gauge group, for the representation of the scalar field ϕ . Then the supersymmetric gauge transformation of the superfield V , is given by:

$$\exp(V') = \exp(-ig\Lambda^\dagger) \exp(V) \exp(ig\Lambda) \quad (\text{IV.61})$$

with:

$$V_{ij} = T_{ij}^a V_a \quad (\text{IV.62})$$

$$\Lambda_{ij} = T_{ij}^a \Lambda_a \quad (\text{IV.63})$$

with T_{ij} being the generators for the adjoint representation.

Equation (IV.61), correctly generates the transformation of the vector field:

$$V \rightarrow V' = V + ig(\Lambda - \Lambda^\dagger) \quad (\text{IV.64})$$

As before this gauge transformation can be used for a choice of the Wess - Zumino gauge with only the fields v_m^a , λ^a and D^a . Before turning to the construction of supersymmetric renormalizable lagrangians in terms of vector and scalar superfields, it is worth mentioning that the condition (IV.53) which was used in defining the vector superfield, is perfectly justified by rotating in superspace any of its components with the operator $\exp(\theta Q + \bar{\theta} \bar{Q})$, which would effectively yield the aforementioned condition.

Having defined and analyzed the chiral and vector superfields, an attempt will be made to derive the supersymmetric form for the interaction of chiral superfields and show how Yukawa interactions are related to the interactions amongst scalar fields.

It is known that the equations of motion remain invariant under the transformation: $L' = L + \nabla \Lambda(x)$. Consequently, the construction of an action invariant under supersymmetry, would necessitate a lagrangian which transforms, at most into space derivatives under the action of the supersymmetry. The action formed by such a lagrangian will - by Gauss's theorem - be invariant under supersymmetry transformations.

As was already demonstrated, in superspace, a supersymmetry transformation, is given by: $\exp\{\theta Q + \bar{\theta} \bar{Q}\}$, combining that with the fact that, as was mentioned before, a chiral supermultiplet contains the fields A , ψ and F with mass dimensions: 1, $3/2$, and 2, respectively - since they are related by successive operations of Q ($1/2$) - it becomes obvious that the superspace coordinates θ and $\bar{\theta}$, have dimensions $-1/2$. Using this fact, the dimensions of the component fields in the chiral superfield ϕ , become manifest since it holds: $\phi = A(y) + \sqrt{2} \theta \psi(y) + \theta \theta F(y)$. Using (IV.23) and (IV.24), it becomes clear that the F term transforms into a derivative under a supersymmetry transformation ($\xi Q + \bar{\xi} \bar{Q}$), because the non derivative terms in Q and $\bar{Q}$ lower the power of θ , and therefore do not contribute to δF . This will always be the case for the component field of highest dimension in any given multiplet.

In effect, this signifies a prescription for constructing renormalizable supersymmetric lagrangians: It entails the construction of a general superfield of dimension consistent with renormalisability, and the subsequent projection of its component field of highest dimension. This will transform into a spacetime derivative under supersymmetry, and - as a result - be qualified, to be used as a Lagrangian density. The interaction part of this Lagrangian, is expected to be constructed through the consideration of products of chiral

superfields: $\phi_1, \phi_2, \dots, \phi_n$. It is obvious that $\bar{D}_\alpha(\phi_1 \phi_2 \dots \phi_n) = 0$, since each field is separately annihilated by $\bar{D}_\alpha$. Consequently, the product of any number of chiral superfields, is itself a chiral superfield with general form, given by eq.(IV.42), and (IV.50). As a result, the term of highest dimension is the $\theta\theta$ component, which will transform into spacetime derivatives under a supersymmetry transformation. Consideration of the mass dimensions of ϕ_i and θ , reveals that only products up to cubic terms in ϕ_i , will allow construction of a renormalizable Lagrangian, with dimension not greater than four. In the standard conventional terminology, the relevant $\theta\theta$ component of a product of chiral superfields, which yields a renormalizable interaction Lagrangian, is described as the "F term". Thus:

$$L_i = [\lambda_i \phi_i + \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{3} g_{ijk} \phi_i \phi_j \phi_k]_F \quad (IV.65)$$

where the $\theta\theta$ and the $\bar{\theta}\bar{\theta}$ components are denoted by the subscripts F and $\bar{F}$ respectively.

Having accomplished that, the next step is to consider products of left-handed chiral superfields and their adjoints (right handed chiral superfields).

Since $\phi_j^\dagger$ does not satisfy the condition $\bar{D}_\alpha \phi_j^\dagger = 0$ it is obvious that $\phi_i \phi_j^\dagger$ does not either and as a result it is not a chiral superfield.

$$\begin{aligned}
\phi_i^\dagger(y^\dagger)\phi_j(y) &= A_i^*(y^\dagger)A_j(y) + \sqrt{2} \theta\psi_j(y)A_i^*(y^\dagger) \\
&+ \sqrt{2} \bar{\theta}\bar{\psi}_i(y^\dagger)A_j(y) + \theta\theta A_i^*(y^\dagger)F_j(y) + \bar{\theta}\bar{\theta}F_i^*(y^\dagger)A_j(y) \\
&+ \sqrt{2} \theta\theta\bar{\theta}F_j(y)\bar{\psi}_i(y^\dagger) \\
&+ \sqrt{2} \bar{\theta}\bar{\theta}\theta F_i^*(y^\dagger)\psi_j(y) + \theta\theta\bar{\theta}\bar{\theta}F_i^*(y^\dagger)F_j(y)
\end{aligned} \tag{IV.66}$$

This time the $\theta\theta\bar{\theta}\bar{\theta}$ component is the one with highest dimension and transforms into space derivatives. The term $\phi_i^\dagger\phi_j$, has dimension 2 and $\theta\theta\bar{\theta}\bar{\theta}$ has dimension -2, and consequently the coefficient $\theta\theta\bar{\theta}\bar{\theta}$ has dimension 4 (always in mass units). As a result the term $\phi_i^\dagger\phi_j$ is the only one of its kind, allowed for the construction of a renormalizable Lagrangian. Expanding the fields about x , using eq.(IV.50), and collecting the $\theta\theta\bar{\theta}\bar{\theta}$ terms yields:

$$\begin{aligned}
[\phi_i^\dagger\phi_j]_D &= [F_i^*(x)F_j(x) + \frac{1}{4}A_i^*(x)\square A_j(x) + \frac{1}{4}\square A_i^*(x)A_j(x) \\
&- \frac{1}{2}\partial_m A_i^*\partial^m A_j + i\frac{1}{2}\partial_m\bar{\psi}_i\bar{\sigma}^m\psi_j - i\frac{1}{2}\bar{\psi}_i\sigma^m\partial_m\psi_j]
\end{aligned} \tag{IV.67}$$

where the $\theta\theta\bar{\theta}\bar{\theta}$ component is denoted by the subscript D.

Obviously, the terms involving derivatives are kinetic energy terms, and for a physically reasonable theory, should be diagonal in the indices $i, j, \dots$. Consequently, the supersymmetric form of the kinetic energy term assumes the form:

$$L_{kin.} = [\phi_i^\dagger \phi_i]_D \quad (IV.68)$$

In terms of the component fields the full Lagrangian is:

$$L = L_{kin.} + L_i = i\partial_m \bar{\psi}_i \sigma^m \psi_i + A_i^* \square A_i + F_i^* F_i + [\lambda_i F_i + m_{ij} (A_i F_j - \frac{1}{2} \psi_i \psi_j) + g_{ijk} (A_i A_j F_k - \psi_i \psi_j A_k) + H. C.] \quad (IV.69)$$

In this equation the products of fermion fields, are Lorentz invariant products $\psi^a \psi_a$, in addition all total derivatives, have been dropped.

The auxiliary fields F_i have no kinetic energy term and are eliminated through use of the equations of motion:

$$\frac{\partial L}{\partial F_k^*} = F_k + \lambda_k^* + m_{ik}^* A_i^* + g_{ijk}^* A_i^* A_j^* = 0 \quad (IV.70a)$$

$$\frac{\partial L}{\partial F_k} = F_k^* + \lambda_k + m_{ik} A_i + g_{ijk} A_i A_j = 0 \quad (IV.70b)$$

Using these equations to eliminate F_k and F_k^* , yields:

$$L = i\partial_m \bar{\psi}_i \sigma^m \psi_i + A_i^* \square A_i - \frac{1}{2} m_{ik} \psi_i \psi_k - \frac{1}{2} m_{ik}^* \bar{\psi}_i \bar{\psi}_k - g_{ijk} \psi_i \psi_j A_k - g_{ijk}^* \bar{\psi}_i \bar{\psi}_j A_k^* - V(A_i, A_i^*) \quad (IV.71)$$

In order to comply with the standard supersymmetric terminology, it is worth introducing the concept of the superpotential P , defined by:

$$P = \lambda_i \phi_i + \frac{1}{2} m_{ij} \phi_i \phi_j + \frac{1}{3} g_{ijk} \phi_i \phi_j \phi_k \quad (\text{IV.73})$$

Obviously then, the interaction Lagrangian, is expressed by the F term of P as in eq.(IV.65).

The generalization of the full Lagrangian, introduced above, to superfields transforming non-trivially under a gauge group is straightforward. In this case P can be derived from the most general gauge invariant terms, involving products of chiral superfields, which can be formed. For a renormalizable theory, the superpotential P is bound to have dimension in mass units no greater than 3, which implies that it can only involve products of up to three superfields. Since it obviously cannot mix left handed and right handed chiral superfields, it is conventionally expressed, in terms of left handed chiral fields, as in (IV.73).

The final step in the construction of supersymmetric renormalizable Lagrangians, will deal with the relevant generalization of the Lagrangian for a free vector field. In view of this task, terms which transform into space time derivatives, under supersymmetry transformations, are again of interest. As was already mentioned, in the Wess-Zumino gauge, the vector superfield V is expressed by eq.(IV.58), in terms of the component fields v_m , λ and D. Since λ and D are gauge invariant, a gauge invariant superfield, W_α , involving these components can be constructed by applying the operator $\exp(\theta Q + \bar{\theta} \bar{Q})$ to the component, λ , of highest

dimension. An alternative approach, is to use the covariant operators D and $\bar{D}$, defined in eq.(IV.25) and eq.(IV.27), to project λ_a from V :

$$W_a = -\frac{1}{4} \bar{D} D D_a V \quad (\text{IV.75a})$$

$$\bar{W}_a = -\frac{1}{4} D \bar{D} \bar{D}_a V \quad (\text{IV.75b})$$

By their structure D and $\bar{D}$, satisfy the anticommutation relations:

$$\{D_a, \bar{D}_a\} = -2i\sigma^m_{aa} \partial_m \quad (\text{IV.76a})$$

$$\{D_a, D_\beta\} = \{\bar{D}_a, \bar{D}_\beta\} = 0 \quad (\text{IV.76b})$$

$$\{D_a, Q_\beta\} = \{D_a, \bar{Q}_\beta\} = \{\bar{D}_a, Q_\beta\} = \{\bar{D}_a, \bar{Q}_\beta\} = 0 \quad (\text{IV.76c})$$

This implies that products involving more than two D or $\bar{D}$ terms vanish. Thus it immediately becomes obvious that:

$$\bar{D}_\beta W_a = 0 \quad (\text{IV.77a})$$

$$D_\beta \bar{W}_a = 0 \quad (\text{IV.77b})$$

Apparently then, the fields W_a and $\bar{W}_a$, are left handed and right handed chiral superfields respectively. They are also gauge invariant, for a combination of eq.(IV.76a) with a

gauge transformation, yields:

$$\begin{aligned}
 W_a &\rightarrow -\frac{1}{4} \bar{D} D D_a (V + \phi + \phi^\dagger) \\
 &= W_a - \frac{1}{4} \bar{D} \{ \bar{D}, D_a \} \phi = W_a - \frac{1}{4} \bar{D} 2i \sigma_{aa}^m \partial_m \phi = W_a
 \end{aligned} \tag{IV.78}$$

Rewriting V in terms of the variables $y = x + i\theta\sigma\bar{\theta}$ or $y^\dagger = x - i\theta\sigma\bar{\theta}$, and using the definitions for D and $\bar{D}$, allows computation of W_a and $\bar{W}_a$:

$$\begin{aligned}
 W_a &= -i\lambda_a(y) + \{ \delta_a^\beta D(y) - i\frac{1}{2} [\sigma^m \bar{\sigma}^n]^\beta_a [\partial_m v_n(y) - \partial_n v_m(y)] \} \theta_\beta \\
 &\quad + \theta \theta \sigma_{aa}^m \partial_m \bar{\lambda}^a(y)
 \end{aligned} \tag{IV.79a}$$

$$\bar{W}_a = i\bar{\lambda}_a(y^\dagger)$$

$$\begin{aligned}
 &+ \{ \epsilon_{a\beta} D(y^\dagger) + \frac{1}{2} \epsilon_{a\beta} [\bar{\sigma}^m \sigma^n]^\gamma_\beta [\partial_m v_n(y^\dagger) - \partial_n v_m(y^\dagger)] \} \bar{\theta}^\beta - \epsilon_{a\beta} \bar{\theta} \theta \sigma^{m\beta a} \partial_m \lambda_a(y^\dagger)
 \end{aligned} \tag{IV.79b}$$

The fields W_a , and $\bar{W}_a$ contain only the gauge invariant fields D , λ_a and the abelian field strength:

$$v_{nm} = \partial_m v_n - \partial_n v_m \tag{IV.80}$$

The constraint relating them is:

$$D^a W_a = \bar{D}_a \bar{W}^a \quad (\text{IV.81})$$

Since they both are chiral superfields, the relevant Lagrangian is to be constructed from them by projecting the $\theta\theta$ coefficient - already denominated as the F component - from Lorentz invariant products of W or $\bar{W}$:

$$[W^a W_a]_F = -2i\lambda\sigma^m \partial_m \bar{\chi} - \frac{1}{2} v^{mn} v_{mn} + D^2 + i\frac{1}{4} v^{mn} v^{lk} \epsilon_{mnlk} \quad (\text{IV.82})$$

This expression clearly contains part of the kinetic energy terms for a vector field and a spinor field.

It becomes obvious that the full kinetic Lagrangian will be:

$$L_{\text{kin.}} = \frac{1}{4} \{ [W^a W_a]_F + [\bar{W}_a \bar{W}^a]_{\bar{F}} \} \quad (\text{IV.83})$$

This constitutes the gauge invariant generalization of the Lagrangian for a free vector field. Dropping terms involving total space time derivatives it may be rewritten as:

$$L_{\text{kin.}} = -\frac{1}{4} v^{mn} v_{mn} - i\lambda\sigma^m \partial_m \bar{\chi} + \frac{1}{2} D^2 \quad (\text{IV.84})$$

The auxiliary field, D, may be eliminated using the equations of motion as was the case in eq.(IV.70) for the F term.

The Lagrangian constructed above is the only independent supersymmetric gauge invariant term of dimension no greater than four that can be formed from the vector field V. In particular a mass term:

$$(V^2)_D = -\frac{1}{2}v_m v^m - \chi\lambda - \overline{\chi}\overline{\lambda} + \frac{1}{2}(M^2 + N^2)$$

$$+ i\frac{1}{2}\chi\sigma^m\partial_m\overline{\chi} - i\frac{1}{2}\overline{\chi}\sigma^m\partial_m\chi + \frac{1}{2}C\Box C + CD \quad (\text{IV.85})$$

would not be gauge invariant, just as the addition of a vector boson mass violates gauge invariance in the non-supersymmetric theory.

This concludes the introduction to the superspace formulation of supersymmetry. In what follows, a temporary return to the physical space time will be made, in order to proceed to an analysis of the $N=1$ Wess-Zumino model from a Quantum Field Theoretical perspective.

V. RENORMALIZATION OF THE SUPERGAUGE INVARIANT LAGRANGIAN

In this chapter, the field theoretical properties of the $N=1$ Wess-Zumino model, in the light of renormalization on a Minkowski space-time manifold, will be studied. The model will be shown to be renormalizable to all orders in perturbation theory, its renormalization program being consistent with invariance under supergauge transformations. The remarkable feature of the cancellations of quadratic divergences, leaving only a common wavefunction renormalization required of the theory, is as well studied and exemplified [ILI.74].

A. THE EXACT SYMMETRY CASE

Commencing with the renormalization program of the theory, use will be made again of the Lagrangian introduced in the second chapter:

$$L = L_0 + L_m + L_g \quad (V.1)$$

slightly modified for the sake of convenience as:

$$L_0 = -\frac{1}{2}(\partial S)^2 - \frac{1}{2}(\partial P)^2 - \frac{1}{2}i(\bar{\chi}\gamma \cdot \partial \chi) + \frac{1}{2}F^2 + \frac{1}{2}G^2 \quad (V.2a)$$

$$L_m = m(FS + GP - \frac{1}{2}i\bar{\chi}\chi) \quad (V.2b)$$

$$L_g = g(FS^2 - FP^2 + 2GSP - i\bar{\chi}\chi S + i\bar{\chi}\gamma_5 \chi P) \quad (V.2c)$$

(with $a=1$), so that the supergauge transformations (III.32) assume the form:

$$\delta S = i\bar{\alpha}\chi \quad (V.3a)$$

$$\delta P = i\bar{\alpha}\gamma_5\chi \quad (V.3b)$$

$$\delta\chi = \partial_\mu (S - \gamma_5 P)\gamma^\mu\alpha + (F + \gamma_5 G)\alpha \quad (V.3c)$$

$$\delta F = i\bar{\alpha}\gamma^\mu\partial_\mu\chi \quad (V.3d)$$

$$\delta G = i\bar{\alpha}\gamma_5\gamma^\mu\partial_\mu\chi \quad (V.3e)$$

As was already mentioned, the action constructed from L is invariant under supergauge transformations. In addition, it is worth drawing attention - once more - to the fact that the auxiliary fields F and G could be eliminated from the Lagrangian, by using their equations of motion, which in the presence of nonlinear couplings, turn out to be:

$$F + mS + g(S^2 - P^2) = 0 \quad (V.4a)$$

$$G + mP + 2gSP = 0 \quad (V.4b)$$

However, a formulation in which both F and G , are treated as fundamental fields, satisfying the equations of motion (V.4), is much more convenient, and consistent with the task

of applying a renormalization program to all orders. The remaining equations of motion are:

$$\square S + mF + 2gFS + 2gGP - i\bar{\chi}\chi = 0 \quad (V.5a)$$

$$\square P + mG - 2gFP + 2gGS + i\bar{\chi}\gamma_5\chi = 0 \quad (V.5b)$$

$$\gamma \cdot \partial \chi + m\chi + 2g(S - \gamma_5 P)\chi = 0 \quad (V.5c)$$

The generalization of the conserved supercurrent (III.34) in the presence of nonlinear couplings is:

$$J_\mu = \gamma^\lambda \partial_\lambda (S - \gamma_5 P) \gamma_\mu \chi + m\gamma_\mu (S - \gamma_5 P)\chi + g\gamma_\mu (S - \gamma_5 P)^2 \chi \quad (V.6a)$$

which can also be written as:

$$J_\mu = \gamma^\lambda \partial_\lambda (S - \gamma_5 P) \gamma_\mu \chi - (F + \gamma_5 G) \gamma^\mu \chi \quad (V.6b)$$

As already mentioned, the absence of cubic or quartic or higher order terms involving the auxiliary fields in a non trivial manner, makes the Lagrangian of the model in question, renormalizable, in the usual sense, namely that all divergences can be absorbed in a redefinition of a finite number of parameters. What is desirable however, is to show that it is possible to renormalize the theory, without introducing more parameters than those already existing in the classical Lagrangian (m, g) , or equivalently,

that the renormalization is consistent with supergauge invariance.

Ward identities

The existence of the conserved current (V.6), also implies the existence of Ward identities amongst the Green functions of the theory. For their derivation, use will be made of the generating functional of these functions $Z[J]$, which is to be constructed by introducing a set of external sources $J(x)$ and adding to the Lagrangian a term:

$$L \rightarrow L + J_S S + J_P P + J_F F + J_G G - i \bar{\eta} \chi \quad (V.7)$$

with the sources $J(x)$ being c-number functions of space-time and $\bar{\eta}$ an anticommuting Majorana spinor, all of them properly introduced to leave (V.7) invariant under supergauge transformations. Specifically, since $\delta L = 0$ under supergauge transformations, (V.7) yields:

$$\delta(L + J_S S + J_P P + J_F F + J_G G - i \bar{\eta} \chi) = 0 \rightarrow \delta(J_S S + J_P P + J_F F + J_G G - i \bar{\eta} \chi) = 0$$

replacing the infinitesimal transformations for the fields as given by eq.(V.3), in this expression, eventually yields:

$$\delta J_S = -i (\partial_\mu \bar{\eta}) \gamma^\mu_\alpha \quad (V.8a)$$

$$\delta J_P = i (\partial_\mu \bar{\eta}) \gamma_5 \gamma^\mu_\alpha \quad (V.8b)$$

$$\delta J_F = i\bar{\eta}\alpha \quad (V.8c)$$

$$\delta J_G = i\bar{\eta}\gamma_5\alpha \quad (V.8d)$$

$$\delta\eta = \partial_\mu (J_F - \gamma_5 J_G) \gamma^\mu \alpha + (J_S + \gamma_5 J_P) \alpha \quad (V.8e)$$

Since in the interaction picture, it holds [LUR.68]:

$$Z[J] = \langle 0 | T \exp \{ i \int d^4x [L_g(\phi_n) + J_{\phi_i} \phi_{in}^i] \} | 0 \rangle$$

- with the repeated dummy index i , signifying a summation over all fields ϕ in the Lagrangian - in combination with the invariance of the action under supergauge transformations, shows that under a change (V.8), $Z[J]$ is invariant, i. e.:

$$\delta Z[J] = 0 \quad (V.9)$$

or using eq.(V.8):

$$\begin{aligned} & -i \int d^4x \frac{\delta Z}{\delta J_S} [\partial_\mu \bar{\eta}(x)] \gamma^\mu + i \int d^4x \frac{\delta Z}{\delta J_P} [\partial_\mu \bar{\eta}(x)] \gamma_5 \gamma^\mu + i \int d^4x \frac{\delta Z}{\delta J_F} \bar{\eta}(x) \\ & + i \int d^4x \frac{\delta Z}{\delta J_G} \bar{\eta}(x) \gamma_5 \\ & - i \int d^4x \frac{\delta Z}{\delta \eta} \{ \partial_\mu [J_F(x) - \gamma_5 J_G(x)] \gamma^\mu + J_S(x) + \gamma_5 J_P(x) \} = 0 \end{aligned} \quad (V.10)$$

Equation (V.10), contains all the Ward identities for the Green functions. As an example, the derivation of one of

them is cited. It is to be advanced, by taking the functional derivatives of eq.(V.10) with respect to $J_S(x)$ and $\bar{\eta}(x)$ and then, setting the external sources to zero. i.

e. :

$$\begin{aligned}
 & -i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_S(x)} [\partial_\mu \bar{\eta}(x)] \gamma^\mu \\
 & +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_P(x)} [\partial_\mu \bar{\eta}(x)] \gamma_5 \gamma_\mu \\
 & +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_F(x)} \bar{\eta}(x) \\
 & +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_G(x)} \bar{\eta}(x) \gamma_5 \\
 & -i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta \eta(x)} \{ \partial_\mu [J_F(x) - \gamma_5 J_G(x)] \gamma^\mu + J_S(x) + \gamma_5 J_P(x) \} \\
 & -i \int d^4x \frac{\delta Z}{\delta \eta(x)} \delta(x-x') = 0
 \end{aligned}$$

which implies:

$$\begin{aligned}
 & -i \int d^4x \frac{\delta^3 Z}{\delta \bar{\eta}(y) \delta J_S(x') \delta J_S(x)} [\partial_\mu \bar{\eta}(x)] \gamma^\mu \\
 & -i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_S(x)} \left[\partial_\mu \frac{\delta \bar{\eta}(x)}{\delta \bar{\eta}(y)} \right] \gamma^\mu \\
 & +i \int d^4x \frac{\delta^3 Z}{\delta \bar{\eta}(y) \delta J_S(x') \delta J_P(x)} [\partial_\mu \bar{\eta}(x)] \gamma_5 \gamma_\mu \\
 & +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_P(x)} \partial_\mu \frac{\delta \bar{\eta}(x)}{\delta \bar{\eta}(y)} \gamma_5 \gamma^\mu
 \end{aligned}$$

$$\begin{aligned}
& +i \int d^4x \frac{\delta^3 Z}{\delta \bar{\eta}(y) \delta J_S(x') \delta J_F(x)} \bar{\eta}(x) \\
& +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_F(x)} \frac{\delta \bar{\eta}(x)}{\delta \bar{\eta}(y)} \\
& +i \int d^4x \frac{\delta^3 Z}{\delta \bar{\eta}(y) \delta J_S(x') \delta J_G(x)} \bar{\eta}(x) \gamma_5 \\
& +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_G(x)} \frac{\delta \bar{\eta}(x)}{\delta \bar{\eta}(y)} \gamma_5 \\
& -i \int d^4x \frac{\delta^3 Z}{\delta \bar{\eta}(y) \delta J_S(x') \delta \eta(x)} \{ \partial_\mu [J_F(x) - \gamma_5 J_G(x)] \gamma^\mu + J_S(x) + \gamma_5 J_P(x) \} \\
& -i \int d^4x \frac{\delta^2 Z}{\delta \bar{\eta}(y) \delta \eta(x)} \delta(x-x') = 0
\end{aligned}$$

with $J_S = J_P = J_F = J_G = \eta = 0$. As a result, all third order derivatives in the previous equation vanish. Then since:

$$\frac{\delta \bar{\eta}(x)}{\delta \bar{\eta}(y)} = \delta(x-y)$$

the result is:

$$\begin{aligned}
& -i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_S(x)} \gamma \cdot \partial \delta(x-y) \\
& +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_P(x)} \gamma_5 \gamma \cdot \partial \delta(x-y) \\
& +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_F(x)} \delta(x-y) \\
& +i \int d^4x \frac{\delta^2 Z}{\delta J_S(x') \delta J_G(x)} \gamma_5 \delta(x-y)
\end{aligned}$$

$$-i \int d^4x \frac{\delta^2 Z}{\delta \bar{\eta}(y) \delta \eta(x)} \delta(x-y) = 0$$

Recalling that:

$$\int dx [\delta'(x)] f(x) = - \int dx [f'(x)] \delta(x)$$

the above expression is recast to the following form:

$$\begin{aligned} & i \int d^4x \left[\gamma \cdot \partial \frac{\delta^2 Z}{\delta J_S(x') \delta J_S(x)} \right] \delta(x-y) \\ & - i \int d^4x \left[\gamma_5 \gamma \cdot \partial \frac{\delta^2 Z}{\delta J_S(x') \delta J_P(x)} \right] \delta(x-y) \\ & + i \int d^4x \left[\frac{\delta^2 Z}{\delta J_S(x') \delta J_F(x)} \right] \delta(x-y) \\ & + i \int d^4x \left[\gamma_5 \frac{\delta^2 Z}{\delta J_S(x') \delta J_G(x)} \right] \delta(x-y) \\ & - i \int d^4x \left[\frac{\delta^2 Z}{\delta \bar{\eta}(y) \delta \eta(x)} \right] \delta(x-y) = 0 \end{aligned}$$

whence:

$$i \gamma \cdot \partial \langle 0 | T[S(x)S(y)] | 0 \rangle - i \gamma_5 \gamma \cdot \partial \langle 0 | T[S(x)P(y)] | 0 \rangle$$

$$+ i \langle 0 | T[S(x)F(y)] | 0 \rangle + i \gamma_5 \langle 0 | T[S(x)G(y)] | 0 \rangle$$

$$- i \langle 0 | T[\bar{\chi}(x)\chi(y)] | 0 \rangle = 0$$

However, the connected two point function Γ_{sp} for proper vertices, turns out to be zero, as can easily be derived

from the structure of the Lagrangian. For example, for the one loop proper diagram:

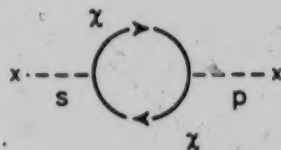


fig. V.1

it holds:

$$I_{SP}(p^2) = g^2 \int d^4k \text{Tr} \left[\frac{\not{k} + m}{k^2 - m^2} \gamma_5 \frac{\not{p} - \not{k} + m}{(p-k)^2 - m^2} \right]$$

$$= g^2 \int d^4k \left\{ \frac{1}{(k^2 - m^2)[(p-k)^2 - m^2]} \text{Tr}[(\not{k} + m) \gamma_5 (\not{p} - \not{k} + m)] \right\} = 0$$

where:

$$\Gamma_{SP}(p^2) = p^2 - m^2 - I_{SP}(p^2)$$

By the same token, the very structure of the Lagrangian, results in a vanishing value for the proper function Γ_{SG} . The vanishing of both of these proper functions, is essentially due to parity conservation of the Lagrangian. Both of the fields P and G have odd intrinsic parity, whereas P and S are characterized by even parity. Consequently the relevant transitions cannot be generated. Eventually then, the whole process, results in the expression:

$$iy \cdot \partial \langle 0 | T[S(x)S(y)] | 0 \rangle + i \langle 0 | T[S(x)F(y)] | 0 \rangle$$

$$-i \langle 0 | T[\bar{\chi}(x)\chi(y)] | 0 \rangle = 0 \quad (V.11)$$

This is the desired Ward identity.

The generating functional for the connected Green functions, is defined to be:

$$W[J] = -i \ln Z[J] \quad (V.12)$$

Since in the interaction picture the generating functional $Z[J]$ is expressed as:

$$Z[J] = \langle 0 | T \exp[i \int d^4x (L_g + J_\phi \phi)] | 0 \rangle$$

then:

$$\ln Z = \langle 0 | T \int d^4x (L_g + J_\phi \phi) | 0 \rangle$$

and eventually:

$$\delta(\ln Z) = \langle 0 | T \int d^4x \delta(L_g + J_\phi \phi) | 0 \rangle = 0$$

That combined with eq.(V.12) and (V.9), implies that the generating functional $W[J]$, satisfies exactly the same equations (V.10) as $Z[J]$. For the renormalization program, it is useful to express the Ward identities, in terms of proper or one-particle irreducible diagrams, since the integrations over internal momenta may be carried out

*independently in each proper subdiagram of a given diagram.

The corresponding generating functional is obtained from $W[J]$, by a functional Legendre transformation:

$$\Gamma[R] = W[J] - \int d^4x (J_S R_S + J_P R_P + J_F R_F + J_G R_G - i\bar{\eta} R_X) \quad (V.13)$$

with:

$$R_\phi = \frac{\delta W}{\delta J_\phi}, \quad R_X = i \frac{\delta W}{\delta \bar{\eta}}, \quad (\phi = S, P, F, G) \quad (V.14)$$

$$J_\phi = -\frac{\delta \Gamma}{\delta R_\phi}, \quad \eta = -i \frac{\delta \Gamma}{\delta \bar{R}_X} \quad (V.15)$$

In terms of $\Gamma[R]$ the Ward identities (V.10) assume the form:

$$\begin{aligned} & iR_X \frac{\delta \Gamma}{\delta R_S} + i\gamma_5 R_X \frac{\delta \Gamma}{\delta R_P} + i\gamma^\mu \partial_\mu R_X \frac{\delta \Gamma}{\delta R_F} \\ & + i\gamma_5 \gamma^\mu \partial_\mu R_X \frac{\delta \Gamma}{\delta R_G} \\ & + (R_F + \gamma_5 R_G - \partial_\mu R_S \gamma^\mu - \partial_\mu R_P \gamma_5 \gamma^\mu) \frac{\delta \Gamma}{\delta \bar{R}_X} = 0 \end{aligned} \quad (V.16)$$

where for the sake of brevity, the relevant integrations have been omitted from the notation.

This is easy to infer from eq.(V.13), since the Legendre transformation on $W[J]$ - for which $\delta W[J]=0$ would immediately imply that $\delta \Gamma=0$. In effect, $\delta(J_S R_S + J_P R_P + J_G R_G - i\bar{\eta} R_X)=0$. Combining that with the expressions (V.8) for the infinitesimal transformations on the sources, yields the corresponding transformations for the fields

R_ϕ , R_x and eventually eq.(V.16). In short, equation (V.16) implies that the functional $\Gamma[R]$ is invariant:

$$\delta\Gamma[R]=0 \quad (V.17)$$

if the fields R_S , R_P , etc. are transformed in an exactly the same way - expressed by eq.(V.3) - as the original ones S , P , etc.

Regularization

The Ward identities expressed by equations (V.10) and (V.16) have been obtained by formal manipulations of the original Lagrangian, without regard to the divergences of perturbation theory. In order to render meaningful the otherwise formal and divergent expressions, it is important - as a preliminary step - to regularize the perturbative expansion.

The regularization scheme, is to be removed after performing the renormalization subtractions. It is imperative however, that the Ward identities be preserved in the light of regularization, that is to say, one has to introduce a regularization scheme which makes the perturbation series finite, while still respecting the supergauge invariance.

The well known Pauli - Villars technique, can easily be applied to the model in question [WES.b74].

For the sake of brevity however, it would be convenient not

to introduce new fields, but instead, higher derivative terms in L . Since only the restricted supergauge transformations with constant parameters α are of interest, it is obvious that the derivatives of the fields transform like the fields themselves.

Therefore, a term which transforms like the kinetic energy term L_0 is:

$$L_\xi = \xi \left[-\frac{1}{2}(\partial_\mu \square S)^2 - \frac{1}{2}(\partial_\mu \square P)^2 - \frac{1}{2}i \square \bar{\chi} \gamma_\mu \partial^\mu \chi + \frac{1}{2}(\square F)^2 + \frac{1}{2}(\square G)^2 \right] \quad (V.18)$$

Eq.(V.18) is enough to make all diagrams, including the tadpole ones, finite, so that no higher order derivatives need be introduced.

It may be - at this point - worth drawing attention to the fact that in a rigorous sense, this regularization scheme is not trully justified, in that the higher derivatives introduced, result in effect in equations of motion involving higher than second order derivatives. However, it is known that classical systems described by higher order differential equations, typically develop non-causal solutions. Consistency on physical grounds, would therefore necessitate a verification of nonexisting causality violation effects. Nevertheless the regularization scheme in question, suffices at the moment for the renormalization program in progress.

The "corrected" through regularization Lagrangian - now including L_ξ as well - results through the equations of

motion in the regularized propagators, cited in the following table:

Contractions	Unregularized	Regularized	Symbol
$\overbrace{S \quad S}$	$\frac{1}{k^2 - m^2}$	$\frac{1 + \xi k^4}{k^2 (1 + \xi k^4)^2 - m^2}$	-----
$\overbrace{S \quad F}$	$-\frac{m}{k^2 - m^2}$	$-\frac{m}{k^2 (1 + \xi k^4)^2 - m^2}$	----- ~~~~~
$\overbrace{F \quad F}$	$-\frac{k^2}{k^2 - m^2}$	$-\frac{k^2 (1 + \xi k^4)}{k^2 (1 + \xi k^4)^2 - m^2}$	~~~~~
$\overbrace{X \quad X}$	$-\frac{1}{\gamma \cdot k - im}$	$-\frac{1}{\gamma \cdot k (1 + \xi k^4) - im}$	_____
$\overbrace{P \quad P} = \overbrace{S \quad S}$			
$\overbrace{P \quad G} = \overbrace{S \quad F}$			
$\overbrace{G \quad G} = \overbrace{F \quad F}$			

Attention is to be drawn to the unfamiliar structure of the "propagators" involving auxiliary fields. Such expressions are expected, since the latter do not have the same dimensions as the ordinary fields.

It would be desirable, at this point, to derive an expression for the superficial degree of divergence of a given proper graph, in the absence of regularization, which would - as usual - in principle suggest all relevant divergences.

By definition, the number of independent loops (L), is nothing but the number of independent internal four momenta in a Feynman diagram, when conservation laws at each vertex,

have been taken into account. For a connected diagram, with I internal lines and V vertices, there are $V\delta^4$ functions expressing this conservation and after extracting the conservation of the incoming momenta, one is left with $V-1$ constraints. Therefore the number of independent momenta or loops, is:

$$L = I - (V - 1) \quad (V.19)$$

In the present case, I is the number of internal bosonic lines $I_{Bs.}$ and the number of the fermionic ones $I_{Fr.}$ added together, so that eq.(V.19) becomes:

$$L = I_{Bs.} + I_{Fr.} - (V - 1) \quad (V.20)$$

In addition; it obviously holds:

$$I_{Bs.} = I_S + I_F + I_{SF} + I_G + I_P + I_{GP} \quad (V.21a)$$

$$I_{Fr.} = I_X \quad (V.21b)$$

However, the complication arising from the existence of auxiliary fields in this theory, precludes any direct use of the standard formula for the superficial degree of divergence ω of a graph G involving no derivatives, acting on the fields at any of its vertices [ITZ.78]:

$$\omega(G) = 4L - I_{Fr.} - 2I_{Bs.} \quad (V.22)$$

A proper modification of this formula, would first require its combination with eq.(V.20):

$$\omega(G) = 2I_{Bs.} + 3I_{Pr.} + 4 - 4V \quad (V.23)$$

Furthermore, let:

$$V = V_1 + V_2 + V_3 + V_4 + V_5 \quad (V.24)$$

with V_i , $i=1, \dots, 5$ being respectively, the number of the following vertices:

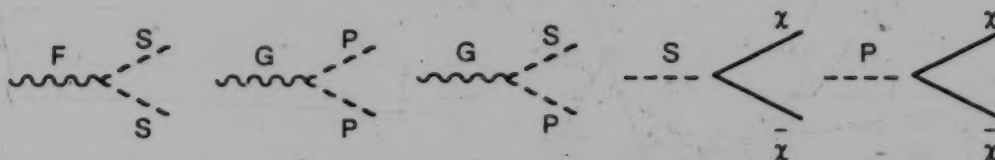


fig. V.2

subsequently, multiplying each one of these three point vertices by a factor of 3, would allow the expressions (in an obvious notation):

$$3V_1 = S + F, \quad 3V_2 = P + G, \quad 3V_3 = G + S + P$$

$$3V_4 = S + \chi, \quad 3V_5 = P + \chi \quad (V.25)$$

so that combining with eq.(V.24), yields:

$$3V = 3S + 3P + F + 2G + 2\chi \quad (V.26)$$

Consideration of the following diagram, would illustrate the previous analysis somewhat more:



fig. V.3

for this it obviously holds: $V_1=2$, $F=2$, $S=4$, according to the notation implicitly introduced above.

Then, the fact that: $3V_1=6$ and $S+F=6$, immediately results in the first of the equations in the set (V.25), namely: $3V_1=S+F$.

Now, considering the vertices given above, as well as equation (V.25), allows for the general expression regarding the number of lines for each field in a given diagram, after having extended the S and P lines in the third vertex to the internal SF and PG, respectively:

$$S=2I_S+I_{SF}+E_S \quad (V.27a)$$

$$P=2I_P+I_{PG}+E_P \quad (V.27b)$$

$$F=2I_F+I_{SF}+E_F \quad (V.27c)$$

$$G=2I_G+I_{PG}+E_G \quad (V.27d)$$

$$x = 2I_x + E_x \quad (V.27e)$$

It is obvious, that the number of vertices of the first type: V_1 , entails 2S lines, V_3 and V_4 entail one, and all others entail no S lines at all. Hence:

$$S = 2V_1 + V_3 + V_4 \quad (V.28a)$$

Likewise:

$$P = 2V_2 + V_3 + V_5 \quad (V.28b)$$

$$F = V_1 \quad (V.28c)$$

$$G = V_2 + V_3 \quad (V.28d)$$

$$x = 2V_4 + 2V_5 \quad (V.28e)$$

The ensuing step, is the combination:

$$\frac{3}{2}x + S + P + 2F + G = 4(V_1 + V_2 + V_3)$$

$$+ 4(V_4 + V_5) = 4V \quad (V.29)$$

where use has been made of equation (V.27)

At the same time, equation (V.28) implies that:

$$\frac{3}{2}X + S + P + 2F + G = \frac{3}{2}E_X + 3I_X$$

$$+E_S + I_{SF} + 2I_S + E_P + I_{PG} + 2I_P$$

$$+2E_F + 4I_F + 2I_{SF} + 2E_G + 4I_G + 2I_{GP} = 4V \quad (V.30)$$

where the last equality apparently refers to eq.(V.29).

Replacing eq.(V.30) in (V.23) and combining with (V.21) eventually gives rise to:

$$d = 4 - I_{SF} - I_{PG} - E_S - E_P - 2E_F - 2E_G - \frac{3}{2}E_X \quad (V.31)$$

which is the desired expression for the superficial degree of divergence - denoted by "d" in this analysis - with I_{SF} and I_{PG} being the numbers of internal SF and PG lines respectively, and E , denoting the number of external lines of the type ϕ .

For $\epsilon \neq 0$, all diagrams are finite. Therefore, the manipulations leading to the Ward identities (V.10) and (V.16) are now permissible, and these identities are indeed satisfied by the unrenormalized but regularized Green functions.

The entire analysis so far, furnishes the liberty required for the pursuit of the main task of this program.

Renormalization

The relevant procedure, commences by tracking down all divergent diagrams to all orders of perturbation theory. The effective degree of divergence is to be found by invoking - in each case - eq.(V.31), and combining with all possible symmetries in effect.

Upon completion of this task, appropriate counterterms will be implicitly introduced. The notation:

$$\Gamma_{\phi_1 \dots \phi_n}(p_1 \dots p_n)$$

will henceforth be used to denote the unrenormalized but regularized n point vertex function with external lines of the type $\phi_1 \dots \phi_n$, calculated through the aforementioned procedure.

For the sake of convenience the renormalization conditions will be set at the origin in momentum space.

Starting with the one point function - which as usual describes the tadpole term - it is easy to see that the fields P , G and χ , have vanishing vacuum expectation values on grounds of parity conservation and Lorentz invariance respectively. However, the scalar character of the fields S and F , does not preclude the eventuality of a nonvanishing vacuum expectation value for these fields. Taking the first derivatives of eq.(V.10) with respect to χ , and letting all sources vanish, immediately yields:

$$\langle 0|F|0\rangle=0 \quad (V.32)$$

At the same time, the equation of motion obtained by varying $L+L_1$ with respect to F reads:

$$(1+\xi\Box)F+mS+g(S^2-P^2)=0 \quad (V.33)$$

Taking the vacuum expectation values and using eq.(V.32) and the Ward identity $\langle 0|T(SS)|0\rangle=\langle 0|T(PP)|0\rangle$, results effectively in:

$$\langle 0|S|0\rangle=0 \quad (V.34)$$

Consequently, all vacuum expectation values vanish, and as a result no counterterms linear in the fields are required. Apparently, only the higher Green functions are important to the renormalization program in progress.

This leaves the two point functions for proper vertices as the subsequent step in this analysis.

Equation (V.31) suggests the existence of four logarithmically divergent functions, namely:

$\Gamma_{FF}(p^2)$, $\Gamma_{GG}(p^2)$, $\Gamma_{GP}(p^2)$, $\Gamma_{FS}(p^2)$ (with $d=0$), two quadratically divergent ones ($d=2$): $\Gamma_{SS}(p^2)$, $\Gamma_{PP}(p^2)$, and it also assigns a value of $d=1$ to the function $\Gamma_{xx}(p)$. Straightforward calculation of the loop integral over internal momenta, confirms the logarithmic divergence for the first four of them and the quadratic one for the next two.

As to the $\Gamma_{xx}(p)$, which at the one loop level represents the graphs:

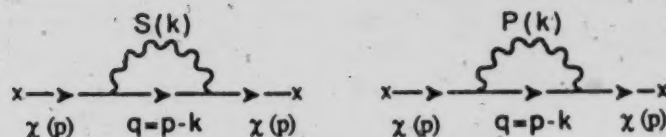


fig. V.4

the integral for the first term (without the regulator ξ) is:

$$I_{xx}(p) = -g^2 \int d^4k \frac{1}{(\not{p}-\not{k})-m} \frac{1}{k^2-m^2}$$

$$= -g^2 \int d^4k \frac{(\not{p}-\not{k})+m}{(\not{p}-\not{k})^2-m^2} \frac{1}{k^2-m^2}$$

or, since the integral over k vanishes:

$$I_{xx}(p) = -g^2 \int d^4k \frac{\not{p}+m}{(\not{p}-\not{k})^2-m^2} \frac{1}{k^2-m^2}$$

with an apparent effective degree of divergence, equal to zero. The same result holds for the other graph as well, so that $\Gamma_{xx}(p)$ is only logarithmically divergent.

It is well known that self energy corrections of a bosonic character to a fermionic propagator, introduce two logarithmically divergent terms (e.g.: Q.E.D. or pseudoscalar theory).

As usual, these two functions are obtained by writing the corresponding Green function in an analytical form, on

grounds of Lorentz invariance:

$$\Gamma_{xx}(p) = \gamma \cdot p \Gamma_{xx}^{(1)}(p^2) - i \Gamma_{xx}^{(2)}(p^2) \quad (V.35)$$

with $\Gamma_{xx}^{(1)}(p^2)$ and $\Gamma_{xx}^{(2)}(p^2)$, both being logarithmically divergent functions, so that eventually - as far as the two point proper functions are concerned - the renormalization program in question necessitates ten counterterms, since $\Gamma_{ss}(p^2)$ and $\Gamma_{pp}(p^2)$, would require two counterterms each, to cancel the quadratic and logarithmic divergences inherent in each one of them.

Obviously, this result holds true for any number of loops, for the simple reason that the effective degree of divergence of a diagram - which eventually determines its divergence - cannot be any higher than the superficial one, and equation (V.31) clearly indicates, that any increase in the number of loops, would not result in an increase of the superficial degree of divergence.

Furthermore, as regards, to the three point functions, (V.31) suggests the existence of as many as eight divergent ones, namely: $\Gamma_{ppp}(p_1, p_2)$ with a superficial degree of divergence $d=1$, characterizing as well the functions $\Gamma_{sss}(p_1, p_2)$, $\Gamma_{spp}(p_1, p_2)$ and five logarithmically divergent functions ($d=0$): $\Gamma_{fss}(p_1, p_2)$, $\Gamma_{fpp}(p_1, p_2)$, $\Gamma_{gsp}(p_1, p_2)$, $\Gamma_{xys}(p_1, p_2)$, $\Gamma_{xyp}(p_1, p_2)$. However, the loop integral for $\Gamma_{ppp}(p_1, p_2)$

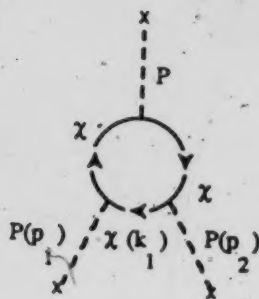


fig. v.5

is:

$$I_{PPP}(p_1, p_2) = ig^3 \int d^4 k_1 \text{Tr}[\gamma_5 S(k_1) \gamma_5 S(k_1 + p_1) \gamma_5 S(k_1 - p_2)] = 0$$

as can easily be confirmed by inspection of the trace.

The underlying implication again, is that the pseudoscalar field P , cannot undergo a transition to two P , as the combination of the relevant intrinsic parities and angular momenta would eventually violate the total parity. As to the $I_{SSS}(p_1, p_2)$ it holds:

$$I_{SSS}(p_1, p_2) = ig^3 \int d^4 k_1 \text{Tr}[S(p_1) S(p_2) S(p_3)]$$

with its trace being the expression:

$$\text{Tr} \{ [(k_1^2 - m^2)[(k_1 + p_1)^2 - m^2][(k_1 - p_2)^2 - m^2]]^{-1}$$

$$\cdot (-\not{k}_1 \not{p}_1 \not{p}_2 + k_1^2 \not{p}_1 + m \not{k}_1 \not{p}_1 - k_1^2 \not{p}_2 + k_1^2 \not{k}_1$$

$$+ m k_1^2 - 2m \not{k}_1 \not{p}_2 + 2m k_1^2 + 2m^2 \not{k}_1 - m \not{p}_1 \not{p}_2 + m \not{p}_1 \not{k}_1 + m^2 \not{p}_1) \}$$

Since:

$$\begin{aligned}\text{Tr}(k_1 \not{p}_1 \not{p}_2) &= \text{Tr}(k_1^2 \not{p}_1) = \text{Tr}(k_1^2 \not{p}_2) \\ &= \text{Tr}(k_1^2 k_1) = \text{Tr}(2m^2 k_1) = \text{Tr}(m^2 \not{p}_1) = 0\end{aligned}$$

the highest power of k_1 in the trace, turns out to be two, so that by inspection, the effective degree of divergence of the function $\Gamma_{SS}(p_1, p_2)$ is zero, signifying only a logarithmic divergence. Likewise the function $\Gamma_{SP}(p_1, p_2)$ is also logarithmically divergent.

It is a straightforward matter to confirm the logarithmic divergence of the five remaining three point functions. In effect, the analysis, so far, has resulted in only seven (logarithmically divergent) three point functions. That each one of them, is to be disposed of, by only one counterterm, is rather easy to perceive. Consideration - for instance - of the function: $\Gamma_{XS}(p_1, p_2)$, would reveal that - on grounds of Lorentz invariance - it assumes the analytical form:

$$\begin{aligned}\Gamma_{XS}(p_1, p_2) &= \not{p}_1 \Gamma_{XS}^{(1)}(p_1^2, p_2^2) + \not{p}_2 \Gamma_{XS}^{(2)}(p_1^2, p_2^2) \\ &+ \Gamma_{XS}^{(3)}(p_1^2, p_2^2) + \Gamma_{XS}^{(\text{fin.})}(p_1^2, p_2^2)\end{aligned}\tag{V.36}$$

Regardless of the form of the renormalization conditions, reference to which will be made in due time, the fact that they are to be set at the origin in momentum space: $p_1 = p_2 = 0$, implies that the first two terms in equation (V.36), vanish,

leaving the logarithmic divergence entirely on $\Gamma_{xxS}^{(3)}(0,0)$. This reasoning does not obviously hold true in the corresponding case of the two point functions, like that expressed by equation (V.35), where use is made of two renormalization conditions, instead of one, invariably used for the three point or higher order proper functions.

As was the case with the two point functions, the aforementioned analysis for the three-point ones, implicitly applied to the one loop level, immediately generalizes to all orders on account of the equation (V.31)

Finally, by invoking once more eq.(V.31), and retracing the same steps as before, it is found that the theory in question, has three logarithmically divergent four point functions, namely the:

$\Gamma_{SSSS}(p_1, p_2, p_3)$, $\Gamma_{PPPP}(p_1, p_2, p_3)$, $\Gamma_{SSPP}(p_1, p_2, p_3)$. All higher order proper functions are obviously finite.

Summarizing, the preceeding analysis has shown, that the divergences of the theory, can be disposed of, by properly introducing ten counterterms for the two point functions, seven for the three, and three for the four point functions.

In view of this task, the renormalization conditions are first to be introduced. As has already been stressed, these are to be set at the origin in momentum space. The demand to be laid, in principle, is that the renormalized proper functions - effected by the addition of counterterms to the already renormalized ones to second

order, and by the renormalized Lagrangian to all orders - satisfy:

$$\Gamma_{\phi\phi}, (p^2=0)_R = m_{R\phi\phi}, \text{ if } \phi\phi': \text{SF, PG or } \bar{\chi}\chi \quad (\text{V.37a})$$

$$\Gamma_{\phi\phi}, (p^2=0)_R = m_{R\phi\phi}^2, \text{ if } \phi\phi': \text{SS, PP} \quad (\text{V.37b})$$

$$\frac{d}{dp^2} \Gamma_{\phi\phi}, (p^2)_R |_{p^2=0} = 1 \text{ if } \phi\phi': \text{SS, PP} \quad (\text{V.37c})$$

$$\frac{1}{4} \gamma^\mu \frac{d}{dp_\mu} \Gamma_{\chi\chi}(p)_R |_{p=0} = 1 \quad (\text{V.37d})$$

and in addition:

$$\Gamma_{\phi_1\phi_2\phi_3}(0,0)_R = c_{\phi_1\phi_2\phi_3}^{(1)}, \Gamma_{\phi_1\phi_2\phi_3\phi_4}(0,0,0)_R = c_{\phi_1\phi_2\phi_3\phi_4}^{(2)} \quad (\text{V.37e})$$

The significance of the relevant renormalized proper functions with auxiliary fields, pertaining to equations (V.37), is going to be discussed shortly.

In addition, m_R is a parameter with mass dimensions, eventhough it is not the physical mass, and $c^{(1)}$, $c^{(2)}$ are constant parameters which remain to be determined.

As it is known, there is a great amount of arbitrariness in the choice of the renormalization conditions. The only requirement is that they must be satisfied to lowest order, so as to fix unambiguously the subtractions to highest orders. What is desirable at this

point, is to prove that the choice in question, can be made consistent with the Ward identities, which in effect eliminate the related arbitrariness to a great extent.

It is worth noticing, that the relations (V.37), introduce a total of twenty parameters, for the renormalization of the theory, one for any relevant combination of fields ϕ , in each relation. Next taking different derivatives of (V.16), and advancing the analysis generally on the lines of the derivation of (V.10), results in the following relations for the two point functions:

$$\begin{aligned} -\Gamma_{SS}(p^2)_R &= -\Gamma_{PP}(p^2)_R = p^2 \Gamma_{XX}^{(1)}(p^2)_R \\ &= p^2 \Gamma_{FF}(p^2)_R = p^2 \Gamma_{GG}(p^2)_R \end{aligned} \quad (V.38a)$$

$$\Gamma_{SF}(p^2)_R = \Gamma_{PG}(p^2)_R = \Gamma_{XX}^{(2)}(p^2)_R \quad (V.38b)$$

These, obviously allow the choice of the values:

$$\Gamma_{SS}(p^2=0)_R = \Gamma_{PP}(p^2=0)_R = 0 \quad (V.39a)$$

$$\begin{aligned} -\Gamma'_{SS}(p^2=0)_R &= -\Gamma'_{PP}(p^2=0)_R = \Gamma_{XX}^{(1)}(p^2=0)_R \\ &= \Gamma_{FF}(p^2=0)_R = \Gamma_{GG}(p^2=0)_R = 1 \end{aligned} \quad (V.39b)$$

$$\Gamma_{SF}(p^2=0)_R = \Gamma_{XX}^{(2)}(p^2=0)_R = \Gamma_{PG}(p^2=0)_R = m_R \quad (V.39c)$$

Likewise the three point functions relate through:

$$\Gamma_{SSS}(p_1=p_2=0)_R = \Gamma_{SPP}(p_1=p_2=0)_R = 0 \quad (V.40a)$$

$$\Gamma_{FSS}(p_1=p_2=0)_R = -\Gamma_{FPP}(p_1=p_2=0)_R = \Gamma_{GSP}(p_1=p_2=0)_R$$

$$= \Gamma_{XXS}(p_1=p_2=0)_R = \Gamma_{XXP}(p_1=p_2=0)_R = 2g_R \equiv c_1 \quad (V.40b)$$

Finally the four point proper functions are arranged by Ward identities of the form:

$$\gamma \cdot p \Gamma_{XXSS}(p_1, p_2, p_3) \approx \Gamma_{SSSS}(p_1, p_2, p_3), \text{ which allow for:}$$

$$\Gamma_{SSSS}(p_1=p_2=p_3=0)_R = \Gamma_{SSPP}(p_1=p_2=p_3=0)_R$$

$$= \Gamma_{PPPP}(p_1=p_2=p_3=0)_R = 0 \quad (V.41)$$

since $\Gamma_{XXSS}(p_1, p_2, p_3)$ is finite.

The manner in which the proper functions involving auxiliary fields, relate to those mentioned explicitly in the renormalization conditions expressed by equations (V.37), now becomes manifest, through (V.38) and (V.39)

Since the divergences of the four point functions vanish on account of equation (V.41), it is to be expected - upon renormalization - that the coupling constant will be redefined only by the three point functions. This in effect suggests the replacement of the constant $c^{(1)}$ introduced in

(V.37), by the renormalized coupling constant g_R , which remains to be defined.

The preceding analysis, has revealed a remarkable feature of this theory. Supergauge invariance, restricts the number of parameters necessary for renormalization, from twenty to at most three.

Henceforth, the analysis is to be advanced on the assumption that the divergences of the theory, can be eliminated to all orders through the introduction of appropriate counterterms. Generally speaking, it is known that if a proper diagram G , of a renormalizable theory is superficially divergent, the corresponding counterterms have to be introduced with the same number, of external fermionic and bosonic lines, as well as the same number of derivatives acting on any of its vertices, as those determining the superficial degree of divergence of G . Hence, for instance, the two point function - at the one loop level - $\Gamma_{SS}^{(2)}(p^2)$ (unrenormalized but regularized), analytically expressed as:

$$\Gamma_{SS}^{(2)}(p^2) = \Gamma_{SS}^{(2)}(p^2=0) - p^2 [\Gamma_{SS}^{(2)}(p^2=0)]' + \Gamma_{fin}^{(2)}(p^2) \quad (V.42)$$

- with $\Gamma_{SS}^{(2)}(p^2=0)$, being quadratically divergent an expression, when $\xi \rightarrow 0$, $[\Gamma_{SS}^{(2)}(p^2=0)]'$, being logarithmically divergent, and $\Gamma_{fin}^{(2)}(p^2)$ being finite upon removal of regularization - would necessitate two counterterms to dispose of the divergences $\Gamma_{SS}^{(2)}(p^2=0)$, $[\Gamma_{SS}^{(2)}(p^2=0)]'$. The counterterm to second order in the Lagrangian reads:

$$\Delta L^{(2)} = -a_{SS}^{(2)} \frac{1}{2} (\partial S)^2 + b_{SS}^{(2)} S^2 \quad (V.43)$$

hence, giving to:

$$\Gamma_{SS}^{(2)}(p^2)$$

an extra contribution of the form:

$$\Delta \Gamma^{(2)} = -a_{SS}^{(2)} p^2 + b_{SS}^{(2)} \quad (V.44)$$

Now the conditions (V.37), yield to second order:

$$\Gamma_{SS}^{(2)}(p^2=0)_R = \Gamma_{SS}^{(2)}(p^2=0) + b_{SS}^{(2)} = m_R^2 \quad (V.45a)$$

$$-\frac{d}{dp^2} \Gamma_{SS}^{(2)}(p^2)_R|_{p^2=0} = [\Gamma_{SS}^{(2)}(p^2=0)]' + a_{SS}^{(2)} = 1 \quad (V.45b)$$

However, by virtue of (V.39a), (V.45a) - which only in principle holds true - eventually is:

$$\Gamma_{SS}^{(2)}(p^2=0)_R = 0$$

and (V.45b), which conforms with (V.39b) gives:

$$a_{SS}^{(2)} = 1 - [\Gamma_{SS}^{(2)}(p^2=0)]' \quad (V.46)$$

Denoting:

$$1 - [\Gamma_{SS}^{(2)}(p^2=0)]' \equiv z^{(2)} \quad (V.47a)$$

results in:

$$Z^{(2)} = a_{SS}^{(2)} \quad (V.47b)$$

Remaining always at the one loop level and introducing on the same lines counterterms to dispose of the divergences of the rest of the two point functions, and then invoking equations (V.39), would yield:

$$a_{SS}^{(2)} = a_{PP}^{(2)} = a_{XX}^{(2)} = -a_{FF}^{(2)} = -a_{GG}^{(2)} = Z^{(2)} \quad (V.48)$$

$$b_{FS}^{(2)} = b_{XX}^{(2)} = b_{PG}^{(2)} = m_R - m^{(2)} \equiv \delta m^{(2)} \quad (V.49)$$

with:

$$m^{(2)} \equiv \Gamma_{FS}^{(2)}(p^2=0) = \Gamma_{XX}^{(2)}(p=0) \Gamma_{PG}^{(2)}(p^2=0) \quad (V.50)$$

$\delta m^{(2)}$ being the mass counterterm to second order, and $m^{(2)}$, being the bare mass effected from the corrections to the classical Lagrangian, only up to the one loop level.

It has already been mentioned that the counterterm to a divergent proper diagram, has the same number of external bosonic and fermionic lines, as the diagram itself, a number independent of the order of perturbation.

In addition to that, equation (V.31) implies that moving to higher orders, could only decrease the degree of divergence. The preceding analysis, in effect, can be generalized to all orders. The starting point to that effect, will be the general statement based on the principal assumption:

$$\Gamma_{\phi\phi}(p^2)_R = \Gamma_{\phi\phi}(p^2) - \Delta\Gamma_{\phi\phi} \quad (V.51)$$

with $\Gamma_{\phi\phi}$ being the generalized analogue to all orders of equation (V.42), and accordingly:

$$\Delta\Gamma_{\phi\phi} = -a_{\phi\phi}p^2 + b_{\phi\phi} \quad (V.52)$$

in conformity with (V.44).

As a consequence, it follows that:

$$-\frac{d}{dp^2}\Gamma_{\phi\phi}(p^2)_R|_{p^2=0} = \Gamma'_{\phi\phi}(p^2=0) + a_{\phi\phi} = 1 \quad (V.53)$$

Then on the lines of the previous analysis, to second order, equation (V.47) generalizes to:

$$1 - \Gamma'_{\phi\phi}(p^2=0) \equiv Z \quad (V.54)$$

with $\Gamma'_{\phi\phi}(p^2)$ being $\Gamma'_{xx}(1)$ in the case of the xx two point function as suggested by (V.35), and with Z given essentially by:

$$Z = a_{SS} = a_{PP} = a_{xx} = -a_{FF} = -a_{GG} \quad (V.55)$$

as (V.48) suggests.

Apparently now, Z contains the entire logarithmic divergence of the proper function $\Gamma_{\phi\phi}$.

The bare Lagrangian, which is the true Lagrangian of the theory, entailing all divergences, is essentially the

classical Lagrangian (V.2), in the presence of quantum corrections. Having assumed that the divergences of this field theoretical model, can be disposed of to all orders through the introduction of appropriate counterterms, is equivalent to assuming that the bare Lagrangian L_B , can be expressed, as the sum of the renormalized Lagrangian - with all physical parameters finite that is - plus its counterterms.

Hence, for the kinetic term it holds:

$$L_B^0 = -\frac{1}{2}(\partial S_R)^2 - \frac{1}{2}(\partial P_R)^2 - i\frac{1}{2}\bar{\chi}_R \not{\partial} \chi_R$$

$$+ \frac{1}{2}F_R^2 + \frac{1}{2}G_R^2 - \Gamma'_{SS}(p^2=0)\left[-\frac{1}{2}(\partial S_R)^2\right] - \Gamma'_{PP}(p^2=0)\left[-\frac{1}{2}(\partial P_R)^2\right]$$

$$- \Gamma'_{XX}(1)(p^2=0)\left[-i\frac{1}{2}(\bar{\chi} \not{\partial} \chi)\right] - \Gamma'_{FF}(p^2=0)\left(\frac{1}{2}F_R^2\right) - \Gamma'_{GG}(p^2=0)\left(\frac{1}{2}G_R^2\right)$$

or as (V.54) and (V.55) suggest:

$$L_B^0 = -\frac{1}{2}(\partial S_R)^2 - \frac{1}{2}(\partial P_R)^2 - \frac{1}{2}i\bar{\chi}_R \not{\partial} \chi_R + \frac{1}{2}F_R^2 + \frac{1}{2}G_R^2$$

$$+ \frac{1}{2}(Z-1)\left[-(\partial S_R)^2 - (\partial P_R)^2 - \frac{1}{2}i\bar{\chi}_R \not{\partial} \chi_R + \frac{1}{2}F_R^2 + \frac{1}{2}G_R^2\right]$$

or equivalently:

$$L_B^0 = -\frac{1}{2}Z(\partial S_R)^2 - \frac{1}{2}Z(\partial P_R)^2 - \frac{1}{2}Zi\bar{\chi}_R \not{\partial} \chi_R + \frac{1}{2}ZF_R^2 + \frac{1}{2}ZG_R^2 \quad (V.56)$$

where implicit use has been made of the (re)defining relation for the renormalized field

ϕ_R :

$$\phi_R = Z^{-1/2} \phi, \quad \phi = S, P, \chi, F, G \quad (V.57)$$

Then, combining again with (V.56), the original form of the kinetic term of the bare Lagrangian is recovered:

$$L_B^0 = -\frac{1}{2}(\partial S)^2 - \frac{1}{2}(\partial P)^2 - \frac{1}{2}i\bar{\chi}\not{\partial}\chi + \frac{1}{2}F^2 + \frac{1}{2}G^2 \quad (V.58)$$

Henceforth, use will be made of the definition (V.57), in the process of renormalizing the remaining parameters of the theory. Prior to commencing this program however, it is worth recalling that the fact that renormalization amounts to a redefinition of the bare field ϕ to a renormalized one, as expressed by (V.57), implies that unrenormalized and renormalized proper functions, are related through:

$$\Gamma_R^{(n)}(p_1, \dots, p_n, m_R, g_R) = Z^{n/2} \Gamma_{\text{reg.}}^{(n)}(p_1, \dots, p_n, m, g, \xi) \quad (V.59a)$$

and for the connected Green functions:

$$G_R^{(n)}(p_1, \dots, p_n, m_R, g_R) = Z^{-n/2} G_{\text{reg.}}^{(n)}(p_1, \dots, p_n, m, g, \xi) \quad (V.59b)$$

Indeed, the connected renormalized functions, are computed from the Lagrangian L_R , by adding a source term $j\phi_R$, whereas the unrenormalized ones, may be computed from $L_R + \Delta L$ in the presence of regularization, by coupling a source to ϕ , that is $j\phi$. Taking n derivatives with respect to j , leads to the announced relation. The limit $\xi \rightarrow 0$, is understood on the

right-hand side of (V.59).

Finally, proper functions are obtained after elimination of the one particle reducible diagrams, an operation which commutes with renormalization, and amputation with the renormalized or unrenormalized propagator respectively - whence the change in the power of Z .

The aforementioned analysis, allows at this point, the generalization of the mass redefinition (V.49) to all orders. In order to exemplify it, the choice to be made will be the proper function $\Gamma_{xx}(p)$, which to second order describes the graph:



fig. V.6

The assumption that the divergences inherent, in this graph can be disposed of through the introduction of appropriate counterterms, and use of the renormalization conditions, amounts to writing:

$$\begin{aligned}
 Z^{(2)} \Gamma_{xx}(p)^{(2)} &= \cancel{p \Gamma_{xx}^{(1)}(p^2)_R^{(2)}} - \cancel{i \Gamma_{xx}^{(2)}(p^2)_R^{(2)}} \\
 &+ \Gamma_{xx}^f(p^2)_R^{(2)} + p(\Delta^{(1)} \Gamma_{xx})_R^{(2)} + (\Delta^{(2)} \Gamma_{xx})_R^{(2)}
 \end{aligned}
 \tag{V.60}$$

with $\Gamma_{xx}(p)^{(2)}$ being again the regularized but unrenormalized

proper function effected from the bare Lagrangian, with suffix R denoting all parts of the renormalized proper function - effected from the renormalized Lagrangian that is - and with counterterm(s) $(\Delta^{(1)}\Gamma_{xx})_R$ and $(\Delta^{(2)}\Gamma_{xx})_R$ for wavefunction and mass renormalization respectively, being of the same character as the renormalized proper function - with all parameters renormalized.' The existence of the multiplicative factor Z, in front of $\Gamma_{xx}(p)$, stems from equation (V.59a) and the fact that use of the renormalization conditions, would leave the right-hand side of the equation above, entirely finite ($\Gamma_{xx}^{(f)}(p^2)_R^{(2)}$). However, the exact character of Z, will have to be determined order by order.

In view of this task, it has to be noticed that $\Gamma_{xx}^{(2)}(p^2=0)$ in (V.47), is of order g^2 , as including the term $g^2[I^{(2)}(0)]$ with $I^{(2)}$ being the loop integral. As a result $Z^{(2)}$ is of order g^2 . Likewise, $\Gamma_{xx}(p^2)_R^{(2)}$ is obviously of the same order, so that the left-hand side of (V.60) is of order four in g. However the only remaining term in $\Gamma_{fin.}(p^2)_R^{(2)}$, after use of the renormalization conditions, is of order two in g_R , a fact which would render the entire equation, unacceptable, in its exact form. This would impel one to approximate $Z^{(2)}$ to unity:

 'As far as $\Gamma_{xx}^{(1)}(p^2)_R^{(2)}$, and $\Gamma_{xx}^{(2)}(p^2)_R^{(2)}$ are concerned, obviously the first upper index refers to the character of the divergence, whereas the second to the order of the perturbation.

$$z^{(2)} \approx 1$$

(V.61)

in which case, this would easily allow the alteration of equation (V.60) to the usual form:

$$\begin{aligned} \Gamma_{xx}(p^2)^{(2)} = & \not{p} \Gamma_{xx}^{(1)}(p^2)_R^{(2)} - i \Gamma_{xx}^{(2)}(p^2)_R^{(2)} \\ & + \Gamma_{xx}^f(p^2)_R^{(2)} + \not{p}(\Delta^{(1)} \Gamma_{xx})_R^{(2)} + (\Delta^{(2)} \Gamma_{xx})_R^{(2)} \end{aligned} \quad (V.62)$$

The graphical representation of (V.60) (or (V.62)) is cited below:

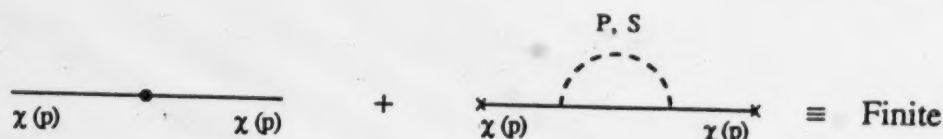


fig. V.7

with all fields entering both graphs being renormalized, and with the mass counterterm, determined by the renormalization conditions, represented by a black blob.

It is worth stressing at this point, that owing to (V.61) the left-hand side of (V.60) is not equal to $\Gamma_{xx}(p)^{(2)}_R$

The next step to this analysis, involves the examination of the proper function $\Gamma_{xx}(p)$ to fourth order.

$\Gamma_{xx}(p)^{(4)}$ represents the following graphs:

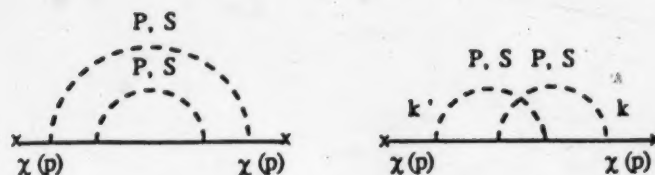


fig. V.8

with a loop integral for the first one being (as $\xi \rightarrow 0$):

$$I^{(4)}(p) = g^4 \int d^4k d^4k' \{ S_F(p-k') S_F(p-k'-k) S_F(p-k') \Delta_F(k) \Delta_F(k') \}$$

or explicitly:

$$I^{(4)}(p) = g^4 \int d^4k d^4k' \left\{ \frac{1}{\not{p} - \not{k}' - m} \frac{1}{\not{p} - \not{k}' - \not{k} - m} \frac{1}{\not{p} - \not{k}' - m} \right. \\ \left. \cdot \frac{1}{k^2 - m^2} \frac{1}{k'^2 - m^2} \right\} \quad (V.63)$$

and accordingly for the other one:

$$I^4(p) = g^4 \int d^4k d^4k' \left\{ \frac{1}{\not{p} - \not{k}' - m} \frac{1}{\not{p} - \not{k}' - \not{k} - m} \frac{1}{\not{p} - \not{k} - m} \right. \\ \left. \cdot \frac{1}{k^2 - m^2} \frac{1}{k'^2 - m^2} \right\} \quad (V.64)$$

By inspection, it becomes obvious that the integral over k in equation (V.63) is linearly divergent. However, as was the case for the $I^{(2)}(p)$, the linear divergence can be shown to disappear upon performing invariant integration, so that the aforementioned integral, is eventually logarithmically divergent. As a result $I^{(4)}(p)$ is logarithmically divergent

in its entirety.

In regard to the integral of equation (V.64), it is easily seen to be already logarithmically divergent with respect to both variables k and k' . Integrals of the type of the two previous ones, are usually handled within the framework of the so called dimensional regularization, a subject reviewed by t'Hooft and Veltman [t'H.72].

As was already proved, equation (V.60), in combination with equation (V.49), as effected by the renormalization conditions [with $m^{(2)} = \Gamma_{xx}(p=0)^{(2)} \equiv \Gamma_{xx}^{(2)}(p^2=0)^{(2)}$], imply that the theory has been renormalized to second order.²

Having accomplished that, the denomination $\Gamma_{xx}(p)_{R_2}^{(4)}$ will be established to represent the unrenormalized (but regularized) proper function to fourth order, which is however renormalized to second order in all respects. Then for the renormalized proper function to fourth order $\Gamma_{xx}(p)_{R_2}^{(4)}$ it holds:

$$\begin{aligned} z^{(4)} \Gamma_{xx}(p)_{R_2}^{(4)} &= p \Gamma_{xx}^{(1)}(p^2)_R^{(4)} - i \Gamma_{xx}^{(2)}(p^2)_R^{(4)} + \Gamma_{xx}^f(p^2)_R^{(4)} \\ &+ p(\Delta^{(1)} \Gamma_{xx})_R^{(4)} + (\Delta^{(2)} \Gamma_{xx})_R^{(4)} \end{aligned} \quad (V.65)$$

This time however, $\Gamma_{xx}(p)_{R_2}^{(4)}$ is of order two in g , a result effected from the - as of yet - unrenormalized loop, whereas all proper functions and counterterms on the right-hand side of (V.65) are of order four in

²It is reminded that the analysis for $\Gamma_{xx}(p)$, holds for all other two point functions of the theory.

g_R . In effect, $Z^{(4)}$ has to be approximated to order two:

$$Z^{(4)} \approx 1 - O(g^2) \quad (V.66)$$

However, the exact nature of $Z^{(4)}$, remains to be determined. Assuming that $\Gamma_{xx}(p)_{R_2}^{(4)}$ relates to the inner loop characterized by momentum k of figure (V.8a), while the outer loop remains unrenormalized, it becomes obvious that since $Z^{(4)}$ is destined to absorb the logarithmic divergence effected by the outer loop, it is essentially of the form:

$$\begin{aligned} Z^{(4)} &= 1 - \Gamma_{xx}^{(1)}(p^2=0)_{R_2}^{(4)} \\ &= -g^2 \gamma_\mu \left\{ \frac{\partial}{\partial p_\mu} \int d^4 k' S_F(p-k') \Delta_F(k') \right\} \Big|_{p=0} \end{aligned} \quad (V.67)$$

Again, the expression $Z^{(4)} \Gamma_{xx}(p)_{R_2}^{(4)}$ on the left-hand side of equation (V.65) is not equal to $\Gamma_{xx}(p)_R^{(4)}$ - as a naive straightforward application of (V.59) would suggest - on account of (V.66).

Now, since $\Gamma_{xx}(p)_{R_2}^{(4)}$ is renormalized to second order, it is bound to satisfy - to that order - an exactly equivalent equation to (V.60):

$$\Gamma_{xx}(p)_{R_2}^{(4)} = \Gamma_{xx}(p)^{(4)} - (\Delta \Gamma_{xx})_{R_2}^{(4)} \quad (V.68)$$

with:

$$\Gamma_{XX}(p)_{R_2} = p\Gamma_{XX}^{(1)}(p^2)_{R_2} - i\Gamma_{XX}^{(2)}(p^2)_{R_2} + \Gamma_{XX}^f(p^2)_{R_2}$$

and:

$$(\Delta\Gamma_{XX})_{R_2} = p(\Delta^{(1)}\Gamma_{XX})_{R_2} + (\Delta^{(2)}\Gamma_{XX})_{R_2}$$

The important fact to observe in eq.(V.68), is that $\Gamma_{XX}(p)^{(4)}$ is of order four in g , as having none of its loops renormalized, whereas $\Gamma_{XX}(p)_{R_2}^{(4)}$ is of order two in g and likewise in g_R , so that through (V.66), eq.(V.68) has to be corrected to:

$$z^{(4)}\Gamma_{XX}(p)_{R_2}^{(4)} = \Gamma_{XX}(p)^{(4)} - (\Delta\Gamma_{XX})_{R_2}^{(4)} \quad (V.69)$$

with $(\Delta\Gamma_{XX})_{R_2}$ introducing two counterterms as was mentioned.

Now a combination of eq.(V.69) with eq.(V.65) results in:

$$\begin{aligned} \Gamma_{XX}(p)^{(4)} &= p\Gamma_{XX}^{(1)}(p^2)_R^{(4)} - i\Gamma_{XX}^{(2)}(p^2)_R^{(4)} + \Gamma_{XX}^f(p^2)_R^{(4)} \\ &+ p(\Delta^{(1)}\Gamma_{XX})_R^{(4)} + (\Delta^{(2)}\Gamma_{XX})_R^{(4)} + p(\Delta^{(1)}\Gamma_{XX})_{R_2}^{(4)} + (\Delta^{(2)}\Gamma_{XX})_{R_2}^{(4)} \end{aligned} \quad (V.70)$$

or:

$$\Gamma_{XX}(p)^{(4)} = \Gamma_{XX}(p)_R^{(4)} + (\Delta\Gamma_{XX})_R^{(4)} + (\Delta\Gamma_{XX})_{R_2}^{(4)} \quad (V.71)$$

The graphical representation of eq.(V.71) follows:

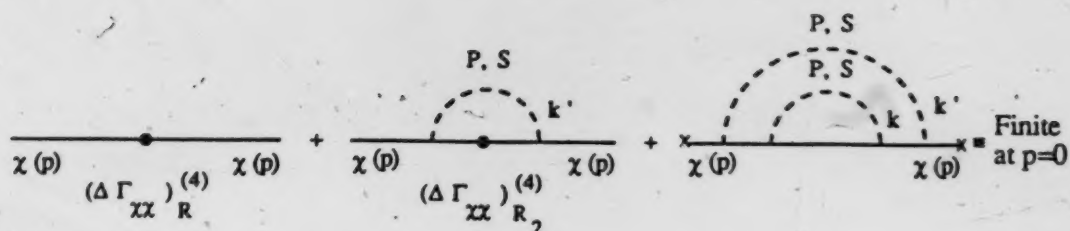


fig. V.9

In the third graph of this figure, the k -integration is made finite by the counterterm $(\Delta\Gamma_{xx})^{(4)}_{R_2}$ of the second graph, leaving the k' -integration to be made finite by the counterterm $(\Delta\Gamma_{xx})^{(4)}_R$ of the first graph. A like statement cannot be made about the following three graphs:



fig. V.10

If the k' -integration in the third graph is performed first, it is made finite by the first graph. However, in order to make the k -integration finite, the second graph is required as well as the first one. This is known as the phenomenon of the overlapping divergences and it is not to be analyzed any further at this point.

The subtlety involved in all this analysis, relates to the number of counterterms in equations (V.70) and (V.65). It is obvious, that the former involves two counterterms[

$(\Delta\Gamma_{xx})_R^{(4)}$ and $(\Delta\Gamma_{xx})_R^{(4)}$ whereas the latter only one: $[p^2(\Delta^{(1)}\Gamma_{xx})_R^{(4)} + (\Delta^{(2)}\Gamma_{xx})_R^{(4)}]$ the other one has been absorbed in $Z^{(4)}$. The net result is that to fourth order, equation (V.65) renders renormalization manifest only at the external loop, as far as graph (a) of fig.(V.8) is concerned, through the corresponding counterterm.

The corresponding expression to equations (V.60) and (V.65), can now readily be written for any order n in the perturbation expansion. On the lines of the derivation of (V.65), it will be of the form:

$$Z^{(n)}\Gamma_{xx}(p)^{(n)} = p\Gamma_{xx}^{(1)}(p^2)_R^{(n)} - i\Gamma_{xx}^{(2)}(p^2)_R^{(n)} + \Gamma_{xx}^f(p^2)_R^{(n)} + p(\Delta^{(1)}\Gamma_{xx})_R^{(n)} + (\Delta^{(2)}\Gamma_{xx})_R^{(n)} \quad (V.72)$$

with $\Gamma_{xx}(p)^{(n)}$ being the regularized two point proper function, renormalized up to order $n-2$, and Z^n being always a logarithmically divergent parameter, approximated to order $n-2$ in g :

$$Z^{(n)} \approx 1 - O(g^{n-2}) \quad (V.73)$$

It is once more stressed at this point, that equation (V.72) was derived order by order, by successively subtracting counterterms and renormalizing. Consequently, it only holds to each specific order and cannot be regarded as having a general validity to all orders. The equation which bares this general validity is - as already mentioned - equation

(V.59).

The expression for the mass term of the bare Lagrangian can now be put forward. It is required to be:

$$L_m^B = m_R (F_R S_R + G_R P_R - \frac{1}{2} i \bar{\chi}_R \chi_R) + (\Delta^{(2)} \Gamma_{FS})_R F_R S_R \\ + (\Delta^{(2)} \Gamma_{GP})_R G_R P_R - (\Delta^{(2)} \Gamma_{XX})_R (\frac{1}{2} i \bar{\chi}_R \chi_R) \quad (V.74)$$

with all fields entering the mass counterterms being renormalized since all proper functions on the right-hand side of eq.(V.72) are renormalized. Then, use of the renormalization conditions (V.37) in (V.72) yields:

$$-\Delta^{(2)} \Gamma_{\phi\phi} = -Z^{(n)} m + m_R, \quad \phi = F, S, G, P, \chi$$

Following standard notation, $\Delta^{(2)} \Gamma_{\phi\phi}$ is replaced by δm so that:

$$m_R = Zm - \delta m \quad (V.75)$$

with the knowledge of the exact character of $Z^{(n)}$, having suggested that the index (n) be dropped.

Combining equation (V.75) with (V.74), eventually results in:

$$L_m^B = m_R (F_R S_R + G_R P_R - \frac{1}{2} i \bar{\chi}_R \chi_R) + (Zm - m_R) (F_R S_R + G_R P_R - \frac{1}{2} i \bar{\chi}_R \chi_R) \\ = Zm (F_R S_R + G_R P_R - \frac{1}{2} i \bar{\chi}_R \chi_R) = m (FS + GP - \frac{1}{2} i \bar{\chi} \chi) \quad (V.76)$$

which is indeed the mass term for the bare Lagrangian.

In order to derive the last step in eq.(V.76), use was made of (V.57)

Obviously now, equation (V.75) through (V.76) implies that the divergences inherent in the unrenormalized two point proper functions, have been absorbed in a redefinition of both the mass and the fields entering the bare Lagrangian to their corresponding renormalized parameters.

This, leaves as the final task for the analysis in progress, the pursuit of the renormalization of the remaining parameter of the theory. The coupling constant renormalization.

This final stage of the renormalization program, necessitates an analysis on the corrections introduced by higher order point proper functions. Since by virtue of equation (V.41), all four point proper functions, are set to zero at the origin in momentum space, when renormalized, all of the aforementioned corrections are introduced by the three point functions. The procedure will be exemplified through the analysis on the $\Gamma_{\chi\chi}(p_1, p_2)$ proper function.

It has already been found, that the divergences of the three point functions inherent in the theory, are all of a logarithmic character. Ignoring those three point functions which - upon renormalization are set equal to zero - by virtue of (V.40a), and since the rest of them when renormalized, are equal through (V.40b), the task ahead is

to isolate their divergent parts, and absorb them in one new parameter, to be eventually introduced in the bare Lagrangian. The relevant to $\Gamma_{\text{xxp}}(p_1, p_2)$ one-particle irreducible diagram pertaining to this process, is to lowest order ($\Gamma_{\text{xxp}}^{(1)}(p_1, p_2)$):

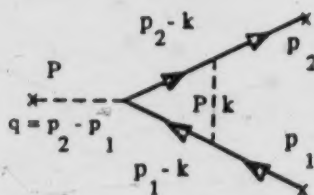


fig. V.11

with $\Gamma_{\text{xxp}}^{(1)}(p_1, p_2)$, giving the first non trivial contribution:

$$(ig)^{-3} \Gamma_{\text{xxp}}^{(1)}(p_1, p_2) = -i \int d^4k [S_F(p_2 - k) \Delta_F(k) S_F(p_1 - k)] \quad (\text{V.77})$$

to $\Gamma_{\text{xxp}}(p_1, p_2)$:

$$\Gamma_{\text{xxp}}(p_1, p_2) = \gamma_5 + \Lambda_5(p_1, p_2) \quad (\text{V.78})$$

with $\Lambda_5(p_1, p_2)$ including contributions to all orders, from all proper trilinear graphs, starting from $\Gamma_{\text{xxp}}^{(1)}(p_1, p_2)$. Since $\Lambda_5(p_1, p_2)$ is a Lorentz pseudoscalar, it is necessarily of the form:

It is reminded that the relation between the truncated three point Green functions and the corresponding proper ones is: $G_{\text{tr}}^{(3)}(x, y, z) = i \Gamma^{(3)}(x, y, z)$

$$\Lambda_5(p_1, p_2) = \gamma_5 f(\not{p}_1 \not{p}_2, p_1^2, p_2^2, p_1 \cdot p_2)$$

with f , some function of the five scalar quantities which can be formed with the aid of the 4-vectors p_1^μ , p_2^μ , γ^μ .

If $\bar{u}(\bar{p}_2)$ and $u(\bar{p}_1)$ are initial and final spinors, then calculating $\langle \bar{u}(\bar{p}_2) | \Lambda_5(p_1, p_2) | u(\bar{p}_1) \rangle$, would allow an expression with all $\not{p}_2$ in f , to the extreme right and all $\not{p}_1$ to the extreme left, setting:

$$\not{p}_1 = \not{m} \text{ with } p_1^2 = m^2 \quad (V.79a)$$

$$\not{p}_2 = \not{m} \text{ with } p_2^2 = m^2 \quad (V.79b)$$

The only remaining variable is $p_1 \cdot p_2$ or, equivalently $(p_1 - p_2)^2$ thus yielding:

$$\bar{u}(\bar{p}_2) \Lambda_5(p_1, p_2) u(\bar{p}_1) = \bar{u}(\bar{p}_1) \gamma_5 u(\bar{p}_2) g[(p_1 - p_2)^2] \quad (V.80)$$

Now setting $g(0) = L$, and performing a subtraction at the origin in momentum space ($p_1 = p_2 = 0$) by writing:

$$\Lambda_5(p_1, p_2) = L \gamma_5 + \Lambda_{5c}(p_1, p_2) \quad (V.81)$$

with $\Lambda_{5c}(p_1, p_2)$, being equal to zero for the conditions expressed by the equations (V.79) as well as for $p_1 = p_2 = 0$, the $L \gamma_5$ term is guaranteed to contain the entire logarithmic divergence of $\Gamma_{\text{xp}}(p_1, p_2)$

Returning now to the one loop calculation, the (infinite) vertex renormalization constant Z' is to be defined through:

$$\Gamma_{\text{xxp}}^{[1]}(p_1, p_2) = -[(Z')^{-1} - 1]\gamma_5 + (Z')^{-1}\Gamma_{\text{xxp}}^{[1]}(p_1, p_2)_R$$

$$= (Z')^{-1}[\Gamma_{\text{xxp}}^{[1]}(p_1, p_2)_R - \gamma_5] + \gamma_5$$

or:

$$Z'(\Gamma_{\text{xxp}}^{[1]}(p_1, p_2) - \gamma_5) = \Gamma_{\text{xxp}}^{[1]}(p_1, p_2)_R - \gamma_5 \rightarrow$$

$$Z'\Lambda_5(p_1, p_2)^{[1]} = \Lambda_5(p_1, p_2)_R^{[1]} \quad (\text{V.82})$$

and combining with the renormalization condition and Ward identity expressed by equation (V.40b) effectively results in:

$$Z'2g = 2g_R$$

or, always being at the one loop level:

$$g_R = Z'g \quad (\text{V.83})$$

Proceeding to the - five vertices entailing - two loop level, it becomes obvious that assuming renormalization for one of them, then the graph:

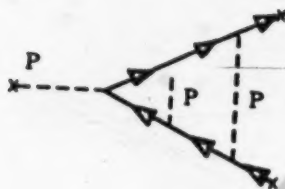


fig. V.12

implies that $\Gamma_{xxP}(p_1, p_2)$ is of order two in g , and $\Gamma_{xxP}(p_1, p_2)_R$ of order five in g_R . Consequently, on the lines of the corresponding derivation for the two point functions, equation (V.82) modifies to:

$$(z^{(5)})^{3/2} z' \Lambda_5(p_1, p_2)^{[5]} = \Lambda_5(p_1, p_2)^{[5]}_R \quad (V.84)$$

with the factor $[z^{(5)}]^{3/2}$ - being of order three in g - accounted for, by (V.59) upon use of the renormalization condition for the three point functions. As was the case with equation (V.82), the parameter z' is determined by (V.84) itself.

As a result of that, equation (V.84), generalizes to any order n as:

$$(z^{(n)})^{3/2} z' \Lambda_5(p_1, p_2)^{[n]} = \Lambda_5(p_1, p_2)^{[n]}_R$$

with $\Lambda_5(p_1, p_2)^{[n]}$, being the corresponding part of the regularized and renormalized up to order $n-2$ three point proper function $\Gamma_{xxP}(p_1, p_2)$, z being always the familiar logarithmically divergent quantity, approximated to order

$n-2$ by equation (V.73), and Z' being determined by the very structure of this equation, is likewise a logarithmically divergent parameter on account of the Ly_5 term.

Dropping the upper indices, the above equation is recast to the form:

$$\Lambda_5(p_1, p_2)_R = [Z]^{3/2} Z' \Lambda_5(p_1, p_2) \quad (V.85)$$

Then use of the renormalization conditions yields again at the origin in momentum space and to all orders:

$$g_R = Z^{3/2} Z' g \quad (V.86)$$

At this point, the exact structure of Z' will have to be determined. Motivated by equation (V.85), one would quite plausibly seek a relation of the form:

$$\Gamma_{XXP}(p_1, p_2) = Z^{-3/2} (Z')^{-1} \Gamma_{XXP}(p_1, p_2)_R \quad (V.87)$$

Now, in order for the claim that equation (V.85) determine the parameter Z' , to be justified, the following choice is made:

$$Z' = 1 - L_R(g_R) \quad (V.88)$$

with $L_R(g_R)$ obviously determined by:

$$\Lambda_5(p_1, p_2)_R = L_R(g_R) \gamma_5 + \Lambda_{5c}(p_1, p_2)_R \quad (V.89)$$

This value for Z' , yields through equation (V.85):

$$\begin{aligned}\Gamma_{\text{XXP}}(p_1, p_2) &= \gamma_5 + \Lambda_5(p_1, p_2) \\ &= \gamma_5 + Z^{-3/2} (Z')^{-1} \Lambda_5(p_1, p_2)_R\end{aligned}$$

Combining with equation (V.89), results in:

$$\begin{aligned}\Gamma_{\text{XXP}}(p_1, p_2) &= \gamma_5 + Z^{-3/2} (Z')^{-1} L_R(g_R) \gamma_5 \\ &\quad + Z^{-3/2} (Z')^{-1} \Lambda_{5c}(p_1, p_2)_R \\ &= Z^{-3/2} [1 - L_R(g_R)]^{-1} (\gamma_5 + \Lambda_{5c}(p_1, p_2)_R) \\ &= Z^{-3/2} (Z')^{-1} \Gamma_{\text{XXP}}(p_1, p_2)_R\end{aligned}$$

where for the last step, use has been made of equation (V.78) in its renormalized form. This result is in agreement with the requirement expressed by (V.87), so that the structure of Z' is indeed determined by the equation (V.88).

Since the counterterms to the renormalized Lagrangian, are determined by the infinite parts of the renormalized proper functions, the counterterm effected by $\Gamma_{\text{XXP}}(p_1, p_2)_R$ will essentially be:

$$Z^{3/2} g_L(g_R) \bar{\chi}_R \chi_R p_R$$

so that generalizing all the preceding analysis, through the appropriate - already familiar - Ward identities, the trilinear sector of the bare Lagrangian is, with respect to

all fields:

$$L_g^B = g_R (F_R S_R^2 - F_R P_R^2 + 2G_R S_R P_R - i\bar{\chi}_R \chi_R S_R + i\bar{\chi}_R \gamma_5 \chi_R P_R) \\ + gZ^{3/2} L_R(g_R) (F_R S_R^2 - F_R P_R^2 + 2G_R S_R P_R - i\bar{\chi}_R \chi_R S_R + i\bar{\chi}_R \gamma_5 \chi_R P_R)$$

and since from equation (V.88): $L_R(g_R) = 1 - Z'$ the above expression eventually results in:

$$L_g^B = g_R (F_R S_R^2 - F_R P_R^2 + 2G_R S_R P_R - i\bar{\chi}_R \chi_R S_R + i\bar{\chi}_R \gamma_5 \chi_R P_R) \\ + Z^{3/2} (1 - Z') g (F_R S_R^2 - F_R P_R^2 + 2G_R S_R P_R - i\bar{\chi}_R \chi_R S_R + i\bar{\chi}_R \gamma_5 \chi_R P_R)$$

or using once more the relation $\phi_R = Z^{-1/2} \phi$ for all fields, along with equation (V.86) would finally give:

$$L_g^B = g (FS^2 - FP^2 + 2GSP - i\bar{\chi}\chi S + i\bar{\chi}\gamma_5 \chi P) \quad (V.90)$$

which is indeed the trilinear term of the bare Lagrangian. Again, the task of absorbing the logarithmic divergence inherent in the unrenormalized three point functions, in the redefinition (V.86) of the physical coupling constant, has been achieved.

At the very inception of this renormalization program, lay the arbitrary assumption, that all divergences, inherent in the theory, can be disposed of through a judicious introduction of counterterms. What is of central importance in the entire procedure, is the fact expressed by the

relations (V.57), (V.75), and (V.86) in combination with those from (V.47) to (V.50). Altogether, they demonstrate the self consistent character of renormalization, manifest in a redefinition of the physical parameters of the theory through absorption of its divergences, which in turn vindicates the principal arbitrary assumption, and concludes the renormalization program of the field theoretical model in question.

The Miraculous Cancellations

In the previous section, it was demonstrated how supergauge invariance manifests itself in renormalization by restricting the parameters necessary for it, from twenty to at most three! The entire analysis has shown that the latter allows for arbitrary mass and coupling constant renormalizations.

At this stage, the task ahead is to explore any further consequences effected by the particular structure of the Lagrangian, which follows not only from supergauge invariance, but also from the requirement of renormalizability and from the particular multiplet of the fields chosen. It turns out that the essential property of L is contained in the identity:

$$\frac{\partial L}{\partial m} = \frac{1}{2g} \frac{\partial L}{\partial S} \quad (V.91)$$

Eq.(V.91) will be used to evaluate the derivative of an arbitrary Green function.

Invoking again the generating functional, allows the expression:

$$\frac{\partial}{\partial m} Z[J] = \frac{\partial}{\partial m} \frac{\int e^{i \int [L+J\phi] d^4x} D[\phi]}{\int e^{i \int d^4x L} D[\phi]} \quad (V.92)$$

where use has been made of the path integral formulation of the generating functional in the interaction picture, with L being the unrenormalized but regularized lagrangian, and with $J\phi$ standing for the source terms in equation (V.7). As usual $D[\phi]$ is assumed to be some invariant measure of integration, and the derivative is with respect to the bare mass, keeping the bare coupling constant g and the regulator parameter ξ fixed.

The derivative of the numerator of eq.(V.92), using eq.(V.91) yields:

$$\begin{aligned} & \frac{\partial}{\partial m} \int e^{i \int [L+J\phi] d^4x} D[\phi] \\ &= i \int e^{i \int [L+J\phi] d^4x} \frac{\partial}{\partial m} \int L_m d^4y D[\phi] \\ &= \frac{i}{2g} \int e^{i \int [L+J\phi] d^4x} \int \frac{\partial}{\partial S} L_g d^4y D[\phi] \\ &= \frac{1}{2g} \int e^{i \int [L-L_g+J\phi] d^4x} \int \frac{\delta}{\delta S(y)} e^{i \int L_g d^4x} d^4y D[\phi] \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2g} \int e^{i/L_g d^4x} \left[\frac{\delta}{\delta S(y)} e^{i/[L-L_g+J\phi] d^4x} d^4y D[\phi] - \right. \\
&= -\frac{im}{2g} \int e^{i/[L+J\phi] d^4x} \left[F d^4y D[\phi] \right. \\
&\quad \left. - \frac{i}{2g} \int e^{i/[L+J\phi] d^4x} \left[J_S d^4y D[\phi] \right. \right. \quad (V.93)
\end{aligned}$$

where an integration by parts has been performed, along with the assumption that the fields fall fast at infinity. At the same time, the derivative of the denominator in eq.(V.92) vanishes because of eq.(V.32). As a result it holds:

$$\frac{\partial}{\partial m} Z[J] = -\frac{im}{2g} \int \frac{\delta Z[J]}{\delta J_F(y)} d^4y - \frac{i}{2g} \int d^4y J_S(y) Z[J] \quad (V.94)$$

The physical significance of eq.(V.94) can be made more transparent if expressed in terms of the generating functional for connected Green functions given by (V.10):

$$\begin{aligned}
\frac{\partial}{\partial m} e^{iW[J]} &= -\frac{im}{2g} \int \frac{\delta}{\delta J_F(y)} e^{iW[J]} d^4y - \frac{i}{2g} \int d^4y J_S(y) e^{iW[J]} \rightarrow \\
\frac{\partial}{\partial m} W[J] &= -\frac{m}{2g} \int \frac{\delta W[J]}{\delta J_F(y)} d^4y - \frac{1}{2g} \int d^4y J_S(y) \quad (V.95)
\end{aligned}$$

From this it is to be deduced, that for every unrenormalized but regularized Green function, other than the vacuum expectation value of S , the derivative with respect to the bare mass, is equivalent to the insertion of a zero momentum F field.

The reason the field F is of zero momentum is rather easy to perceive, since:

$$\begin{aligned} \int F(y) d^4 y &\approx \int \left[\int F(p) e^{ip \cdot y} d^4 p \right] d^4 y \\ &\approx \int F(p) \left[\int e^{ip \cdot y} d^4 y \right] d^4 p \approx \int F(p) \delta^{(4)}(p) d^4 p = F(p=0) \end{aligned}$$

This important property, has an immediate analogue, in terms of the one particle irreducible functions. Invoking the corresponding Legendre transformation expressed by (V.13), eq.(V.95) is expressed as:

$$\frac{\partial \Gamma[R]}{\partial m} = -\frac{m}{2g} \int R_F(y) d^4 y + \frac{1}{2g} \int \frac{\delta \Gamma[R]}{\delta R_S(y)} d^4 y \quad (V.96)$$

which again implies that for every vertex function other than the one particle irreducible part of $\langle 0|F|0 \rangle$, the derivative with respect to the bare mass, gives an insertion of a zero momentum S field.

Use will now be made of the equation (V.96), to derive the first remarkable feature that supersymmetry has on renormalization. Taking the functional derivative with respect to R_F , and then setting all fields R equal to zero, the aforementioned equation yields:

$$\frac{\partial \langle 0|F|0 \rangle}{\partial m} = -\frac{m}{2g} + \frac{1}{2g} \Gamma_{FS}(p^2=0)$$

since $\langle 0|F|0 \rangle = 0$, this eventually yields:

$$m = \Gamma_{FS}(p^2=0)$$

Equation (V.59a) is now to be invoked, in order to recast the previous one in the form:

$$m = Z^{-1} \Gamma_{FS}(p^2=0)_R \quad (V.97)$$

Then, a comparison of equation (V.97) with (V.39c), shows that:

$$m_R = Zm \quad (V.98)$$

One more comparison of (V.98) with (V.75) eventually reveals:

$$\delta m = 0 \quad (V.99)$$

that is no mass renormalization is required!

Furthermore, taking the second derivative of eq.(V.96), with respect to R_x , yields:

$$\frac{\partial}{\partial m} Z^{-1} \Gamma_{xx}(p)_R = \frac{1}{2g} Z^{-3/2} \Gamma_{sxx}(p_S=0, p)_R \quad (V.100)$$

where use has been made of eq.(V.59a)

Then, setting $p=0$ and invoking equations (V.40) and (V.39c) results in:

$$\frac{\partial}{\partial m} Z^{-1} m_R = \frac{1}{2} \frac{1}{Z^{3/2} g} 2g_R$$

or through eq.(V.98):

$$g_R = Z^{3/2} g \quad (V.101)$$

implying that $Z'=1$ in equation (V.86), that is no coupling constant renormalization is required for this theory! As a result, a significant conclusion emerges:

The only counterterm required for the model in question, is merely a common wavefunction renormalization.

In order to illustrate somewhat more, the significance of eq.(V.96) which led to this conclusion, an invocation will be made, of an arbitrary regularized but unrenormalized vertex function $\Gamma_{\dots_n}(p_1 \dots p_n)$ whose only dependence on the bare mass comes from the propagators in the internal lines. As is easily seen, the derivative with respect to the mass of the propagator in an internal line, is equivalent to a mass insertion in it. Explicit calculation of this derivative of all internal lines, using the expressions of table (1), renders the expression $\frac{\partial}{\partial m} \Gamma_{\dots_n}(p_1 \dots p_n)$ in the form of a sum of the diagrams cited in fig. V.13. A subsequent insertion of a zero momentum S field to the vertex function $\Gamma_{\dots_n}(p_1 \dots p_n)$ yields the relevant diagrams for the function $\Gamma_{s, \dots_n}(p_1 \dots p_n)$ illustrated in fig. V.14. Apparently now, equation (V.96) follows by a simple comparison of these diagrams with the previous ones.

The mechanism by which quadratic divergences disappear as eq.(V.99) implies, is the exact cancellation of graphs involving fermion and boson loops. The former come with an extra factor of -1 which allows for the cancellation. In

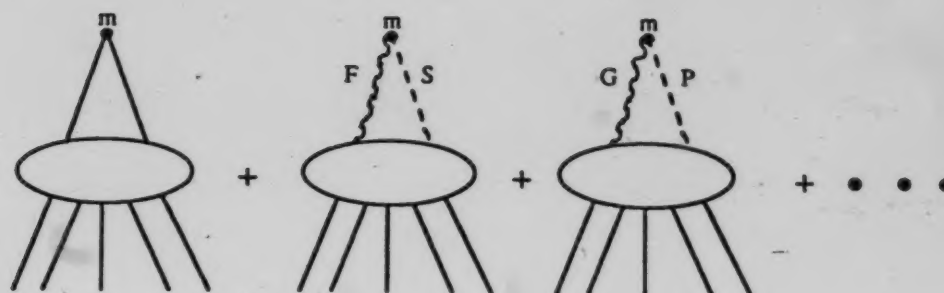


fig. V.13

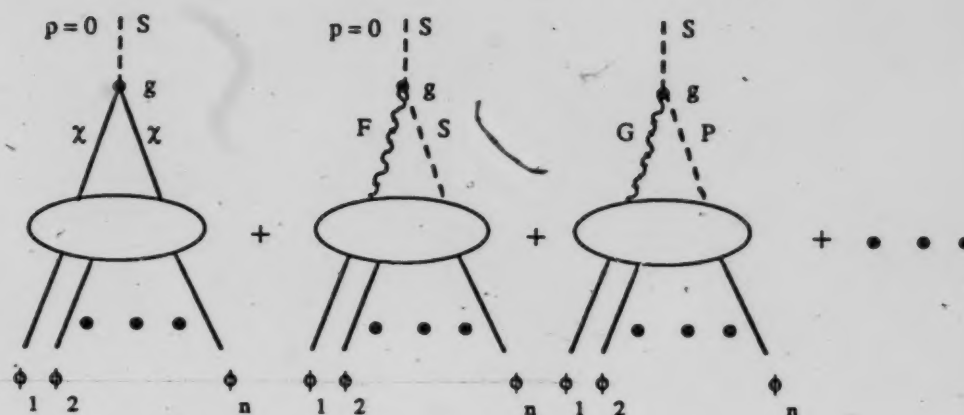


fig. V.14

order to demonstrate that at the one loop level of the model in question, it would prove useful to eliminate the auxiliary fields F and G from the bare Lagrangian through the equations of motion, to end up with:

$$\begin{aligned}
 L^B = & -\frac{1}{2}(\partial_\mu S)^2 - \frac{1}{2}(\partial_\mu P)^2 - \frac{1}{2}i\bar{\chi}\not{\partial}\chi - \frac{1}{2}m^2(S^2 + P^2) - \frac{1}{2}im\bar{\chi}\chi \\
 & - gmS(S^2 + P^2) - \frac{1}{2}g^2(S^2 + P^2)^2 - ig\bar{\chi}(S - i\gamma_5 P)\chi
 \end{aligned}
 \tag{V.102}$$

At one loop, the only potential quadratic divergence can

occur in the self-energies of the scalar fields. The cancellation of quadratic divergences, is to be illustrated in the one loop graph of the S field self-energy. All possible one loop graphs are illustrated in figure V.15.

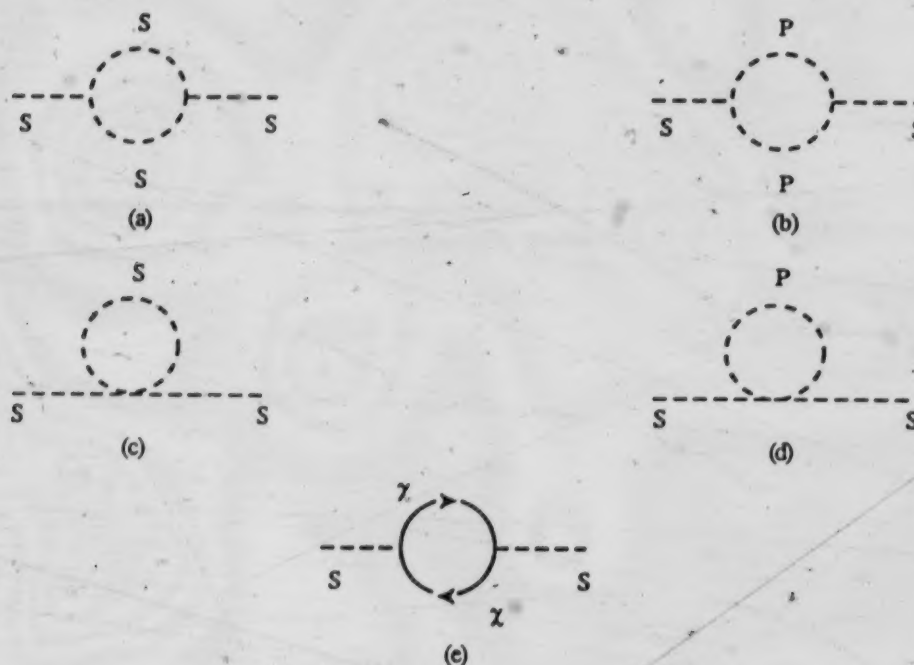


fig. V.15

One loop corrections to the mass of the scalar field S . Graphs (c) and (d), contain quadratic divergences which exactly cancel with (e).

Only graphs (c), (d) and (e) have quadratic divergences as can easily be seen from equation (V.22) (in principle), consequently attention will be focused on those. The sum of

the two boson loop graphs (c) and (d), is the quadratically divergent integral:

$$8g^2 \int \frac{d^4 k}{k^2 - m^2} \quad (V.103)$$

The fermion loop graph (e), is given by:

$$-2g^2 \text{Tr} \int d^4 k \frac{\not{k} + m}{k^2 - m^2} \frac{\not{k} - \not{p} + m}{[(k-p)^2 - m^2]} \quad (V.104)$$

Writing the trace as:

$$\begin{aligned} \text{Tr}(\not{k} + m)(\not{k} - \not{p} + m) &= 4(k^2 - k \cdot p + m^2) \\ &= 2\{(k^2 - m^2) + [(k-p)^2 - m^2] - p^2 + 4m^2\} \end{aligned} \quad (V.105)$$

Inserting this expression in eq.(V.104) results in:

$$-4g^2 \int \frac{d^4 k}{k^2 - m^2} - 4g^2 \int \frac{d^4 k}{(k-p)^2 - m^2} + I(p, m) \quad (V.106)$$

with the integral $I(p, m)$ being only logarithmically divergent.

Imposing the shift: $k \rightarrow k+p$, in the second integral in the expression (V.106), it becomes clear that the first two terms of the same expression, exactly cancel the result given in the expression (V.103). Hence, the quadratic divergence has indeed cancelled

Actually, the aforementioned cancellation constitutes one of

the most central features of supersymmetric theories (where supersymmetry is either exact or broken, spontaneously or by explicit soft terms), namely that the unrenormalized theory contains no quadratic divergences to all orders in perturbation theory. More complicated examples of the cancellation of quadratic divergences in models containing gauge fields have been studied [KAU.a82] [KAU.b82].

This concludes the renormalization of the $N=1$ Wess - Zumino model, in which supergauge invariance is exact, and leaves the analysis in progress, confronted with the final task of exploring the possible implications of the renormalization of this theory with symmetry breaking, which is to be pursued in the remaining part of this chapter.

B. RENORMALIZATION OF THE THEORY WITH SYMMETRY BREAKING

This final section of the renormalization program, involves the analysis of the model in question, in the presence of an explicit symmetry breaking which is sufficiently smooth to permit the use of the same renormalization procedure, as in the symmetric case.

As already mentioned before, if supersymmetric theories are to find realistic applications in high energy physics, supersymmetry must be broken simply on empirical grounds: No experiment has ever resulted in an observation of degenerate BOSE-FERMI multiplets.

Before pursuing the final goal of this program it would be quite instructive to prove that the fermionic nature of the supersymmetric generators, results in a fermionic Goldstone mode when supersymmetry is spontaneously broken. Use will be made of this result in due time.

As is the case with all symmetries based on Lie groups, the relevant procedure commences with the assumption of a conserved current j^μ :

$$\partial_\mu j^\mu = 0 \quad (V.107)$$

Let $A(x)$, be a scalar field in a supersymmetric Lagrangian, then it is known from the theory of conserved currents that:

$$\delta A(x) = -i\bar{\epsilon}_a [Q_a, A(x)] \quad (V.108)$$

so that:

$$\delta \langle 0 | A(x) | 0 \rangle = -i\bar{\epsilon}_a \langle 0 | [Q_a, A(x)] | 0 \rangle \quad (V.109)$$

with Q being the generator of the appropriate symmetry transformation, in this case the supersymmetry generator, and ϵ , the transformation parameter. The spontaneous breakdown of supersymmetry, simply implies that the vacuum is no longer invariant under the corresponding transformations:

$$Q|0\rangle \neq 0 \quad (V.110)$$

which combined with eq.(V.109) implies:

$$\delta \langle 0 | A(x) | 0 \rangle \neq 0 \quad (V.111)$$

In the Weyl formulation then, for any local operator $A(y)$ it holds on account of eq.(V.107):

$$\int_V d^3\bar{x} [\partial_\mu j^\mu(x), A(y)] = 0 \rightarrow$$

$$\frac{d}{dt} \int_V d^3\bar{x} [j^0(\bar{x}, t), A(y)] + \int_S d\bar{s} \cdot [\vec{j}(\bar{x}, t), A(y)] = 0$$

Now, assuming that the fields fall fast enough at infinity, and letting the surface S enclosing the volume V expand indefinitely, so that:

$$[\vec{j}(\bar{x}, t), A(y)] \rightarrow 0$$

eventually gives rise to:

$$\frac{d}{dt} [Q(t), A(y)] = 0 \quad (V.112)$$

with the charge $Q(t)$ being $\int d^3x j^0(x, t)$. It follows that:

$$\frac{d}{dt} \langle 0 | [Q(t), A(y)] | 0 \rangle = 0 \quad (V.113)$$

It is also known that for a function of spacetime $F(x)$, it

*It is reminded here, that when a symmetry is spontaneously broken, its generators being functions of time, are no longer constants of motion, that is no conserved charges exist.

holds; on grounds of Poincare invariance:

$$\partial_\mu F(x) = i[P_\mu, F(x)]$$

and:

$$e^{icP} F(x) e^{-icP} = F(x+c)$$

so that:

$$\begin{aligned} \langle a | \partial_\mu F(x) | b \rangle &= i(p_\mu^a - p_\mu^b) \langle a | F(x) | b \rangle \\ &= i(p_\mu^a - p_\mu^b) \langle a | F(0) | b \rangle \exp[i(p_\mu^a - p_\mu^b)x^\mu] \end{aligned}$$

according to which:

$$\langle 0 | j_0(\bar{x}, t) | p_n \rangle = \langle 0 | j_0(0) | p_n \rangle e^{-ip_n \cdot x} \quad (V.114)$$

Then inserting a complete set of states $|p_n\rangle$, and using (V.114) yields:

$$\begin{aligned} \langle 0 | [Q(t), A(x')] | 0 \rangle &= \int d^3\bar{x} \langle 0 | [j_0(\bar{x}, t), A(x')] | 0 \rangle \\ &= \int d^3\bar{x} \sum_n [\langle 0 | j_0(\bar{x}, t) | p_n \rangle \langle p_n | A(x') | 0 \rangle \\ &\quad - \langle 0 | A(x') | p_n \rangle \langle p_n | j_0(\bar{x}, t) | 0 \rangle] \\ &= \int d^3\bar{x} \sum_n \{ \langle 0 | j_0(0) | p_n \rangle \langle p_n | A(0) | 0 \rangle \exp[-ip_n \cdot (x-x')] \} \end{aligned}$$

$$-\langle 0|A(0)|p_n\rangle\langle p_n|j_0(0)|0\rangle\exp[ip_n\cdot(x-x')]\}$$

$$=\sum_n(2\pi)^3\delta(\bar{p}_n)\cdot\{\langle 0|j_0(0)|p_n\rangle\langle p_n|A(0)|0\rangle\exp[-iE_n(t-t')]\}$$

$$-\langle 0|A(0)|p_n\rangle\langle p_n|j_0(0)|0\rangle\exp[iE_n(t-t')]\}$$

According to which, equation (V.113) implies:

$$\begin{aligned} 0 &= \sum_n (2\pi)^3 \delta(\bar{p}_n) (-iE_n) \cdot \{\langle 0|j_0(0)|p_n\rangle\langle p_n|A(0)|0\rangle\exp[-iE_n(t-t')]\} \\ &+ \langle 0|A(0)|p_n\rangle\langle p_n|j_0(0)|0\rangle\exp[iE_n(t-t')]\} \end{aligned} \quad (V.115)$$

It then follows from (V.109), (V.111) and (V.115) that a state $|p_n\rangle$ must exist such that:

$$\langle 0|j(0)|p_n\rangle\langle p_n|A(0)|0\rangle \neq 0$$

so that from eq.(V.115):

$$E_n \delta(\bar{p}_n) = 0 \text{ with } E_n = (m_n^2 + \bar{p}_n^2)^{1/2}$$

that is the mass of the appropriate particle state must be zero.

Since the condition imposed is that $\langle 0|j_0(0)|p_n\rangle$ be Lorentz invariant, that is:

$$\langle 0|j_0(0)|p_n\rangle = \langle 0|U^\dagger(Uj_0(0)U^\dagger)U|p_n\rangle$$

with U being the relevant Lorentz transformation, the

operator must transform under the same representation of the Lorentz group as the state $|p_n\rangle$. Hence if $j_0(0)$ is a spinor current, $|p_n\rangle$ must be a spinor state. Therefore in a spontaneously broken supersymmetry, the Goldstone mode is a set of massless fermion states, with the same quantum numbers as j_0 and A , since it is generated by these operators from the vacuum. It is conventionally called "Goldstino", in the terminology of supersymmetry.

This then, concludes the proof of the Goldstone theorem for either the Supersymmetric or non-Supersymmetric cases.

Returning to the original Lagrangian, its supergauge invariance becomes broken through the addition of an explicit soft breaking term $L_c = -cS$, with c being a small constant:

$$L = L_0 + L_m + L_g + L_c \quad (V.116)$$

The term L_c breaks supergauge invariance, and the current j^μ is no longer conserved. Instead, use of the equations of motion, in the usual procedure yields:

$$\partial_\mu j^\mu = c\chi \quad (V.117)$$

The linear term in S can be eliminated by a simultaneous shift of the fields S and F : $S \rightarrow S+s$, $F \rightarrow F+f$. Inserting the shifted fields in the original Lagrangian, renders:

$$-\frac{1}{2}[\partial(S+s)]^2 + \frac{1}{2}(F+f)^2 + m(F+f)(S+s) - c(S+s)$$

$$+g(F+f)(S+s)^2 - g(F+f)P^2 + 2gG(S+s)P - g\bar{\chi}\chi(S+s)$$

$$= -\frac{1}{2}(\partial S)^2 + \frac{1}{2}F^2 + mFS - cS$$

$$+gFS^2 - gFP^2 + 2gGSP - ig\bar{\chi}\chi S \rightarrow$$

$$\frac{1}{2}F^2 + Ff + \frac{1}{2}f^2 + mFS + mFs + mfS + mfs - cS + gFS^2 + 2gFSs + gFs^2$$

$$+gFS^2 + 2gFSs - gFP^2 - gFP^2 + gfs^2 + 2gGSP + 2gGsP - ig\bar{\chi}\chi S - ig\bar{\chi}\chi s$$

$$= \frac{1}{2}F^2 + mFS + gFS^2 - gFP^2 + 2gGSP - ig\bar{\chi}\chi S \rightarrow$$

$$Ff + \frac{1}{2}f^2 + mFs + mfS + mfs - cS + 2gFSs + gFs^2$$

$$+gFS^2 + 2gFSs - gFP^2 + gfs^2 + 2gGsP - ig\bar{\chi}\chi s = 0 \rightarrow$$

$$(f+ms+gs^2)F + (mf+2gsf-c)S + \frac{1}{2}f^2 + mfs$$

$$-gFP^2 + gfs^2 + 2gsGP - ig\bar{\chi}\chi s + 2gFSs + gFS^2 = 0$$

Apparently in this expression, all terms expressing couplings of the fields to either of the constants s or f , are bound to vanish. Hence the resulting expression is:

$$(f+ms+gs^2)F + (mf+2gsf-c)S = L(s, f)$$

Requiring the linear terms in F and S to vanish, eventually yields:

$$f+ms+gs^2=0 \quad (V.118)$$

$$mf+2gsf-c=0 \quad (V.119)$$

These two relations, obviously express the conditions that should hold between the constant s and f , so that the linear term in S , be eliminated.

After the shift, the current assumes the form:

$$J^\mu = \gamma^\lambda \partial_\lambda (S - \gamma_5 P) \gamma^\mu \chi - (F + \gamma_5 G) \gamma^\mu \chi - f \gamma^\mu \chi \quad (V.120)$$

and the new form of the Lagrangian reads (apart from a constant):

$$L = L_0 + L_g + (m + 2gs)(FS + GP - \frac{1}{2}i\bar{\chi}\chi) + fg(S^2 - P^2) \quad (V.121)$$

This obviously corresponds to a mass for the χ field:

$$m_\chi = m + 2gs \quad (V.122a)$$

or through eq.(V.119)

$$m_\chi = \frac{c}{f} \quad (V.122b)$$

Use of the equations of motion then, renders the masses of the S and P fields respectively in the form:

$$m_S^2 = m_\chi^2 - 2fg = m_\chi^2 - 2g \frac{c}{m_\chi} \quad (V.123)$$

$$m_p^2 = m_x^2 + 2fg = m_x^2 + 2g \frac{c}{m_x} \quad (V.124)$$

Obviously, unlike the exact symmetry case, the masses are no longer equal. Instead in the tree approximation level they satisfy the relation:

$$m_s^2 + m_p^2 = 2m_x^2 \quad (V.125)$$

Eliminating f in (V.118) and (V.119), yields an equation of third degree in s :

$$s(m+2gs)(m+gs)+c=0 \quad (V.126)$$

For small but finite c , this equation has three solutions. They correspond to the stationarity points of the "potential":

$$V(s) = \frac{1}{2}ms^2 + gms^3 + \frac{1}{2}gs^4 + cs = \frac{1}{2}s^2(m+gs)^2 + cs \quad (V.127)$$

which is obtained by eliminating F and G , subsequently taking the negative of the Lagrangian for $S=s$, $P=x=0$, and it yields the energy density as a function of s . One of the solutions of (V.126) corresponds to the absolute minimum of the potential and it is stable, the other two, to a local maximum and a local minimum respectively and are unstable. Letting $c \rightarrow 0$, the three solutions turn out to be:

$$s_1=0, \quad s_2=-\frac{m}{g}, \quad s_3=-\frac{m}{2g} \quad (V.128)$$

and the potential becomes symmetric about the value s_3 . This time s_1 and s_2 correspond to the two minima of the potential, and are actually two stable, physically equivalent solutions, since they allow a transition from one to the other through a redefinition of the fields. The solution s_3 is unstable as corresponding to a local maximum. This particular solution would yield:

$$m_x = m + 2gs_3 = 0 \quad (V.129)$$

As was proved in the beginning of this section, the spontaneous breakdown of a symmetry, yields a massless (Goldstone) mode. At this point, the conditions $c \rightarrow 0$ (no explicit symmetry breaking) and $s = s_3$ signify a situation in which the Majorana spinor of the theory, turns out to be massless, as equation (V.129) indicates, for which reason through the reverse logic it is deduced that the x field would in this case be a Goldstone spinor and the supergauge symmetry would be spontaneously broken. This however would have the paradoxical result that the masses of the fields S and P would turn out to be given by:

$$m_S^2 = -\frac{1}{2}m^2 \text{ and } m_P^2 = \frac{1}{2}m^2 \quad (V.130)$$

- as it is obvious from eq.(V.123) and (V.124) - so that m_S is imaginary. As it will become apparent in due time, this situation does not change in higher orders. Consequently, the mass term of the S field would be imaginary, and the

$s=s_3$ solution is to be rejected on physical grounds. Nevertheless, it remains an interesting question whether in other supergauge invariant models, a stable spontaneous symmetry breaking can occur.

Since the central property of SUSY, namely:

$$\{Q_i, \bar{Q}_j\} = -2(\sigma)_{ij}{}^\mu P_\mu$$

yields for the Hamiltonian $H=P^0$:

$$H = \frac{1}{4}[Q_i \bar{Q}_i + \bar{Q}_i Q_i + Q_j \bar{Q}_j + \bar{Q}_j Q_j] = \frac{1}{4} \sum_k Q_k^2 \geq 0$$

and since the vacuum expectation value of the left-hand side vanishes, while that of the right-hand side is the sum of four positive terms, it should hold:

$$\langle 0 | Q_k^2 | 0 \rangle = 0$$

whence:

$$Q_k | 0 \rangle = 0$$

that is the supergauge charges always annihilate the vacuum, which indicates that there cannot be any spontaneous breakdown of supergauge symmetry. However, this argument is at the very best heuristic. Indeed, if the symmetry were spontaneously broken, the charges would not exist as operators. It would therefore appear that - as a result - the compelling conclusion is that a spontaneous breakdown in

SUSY is rather unlikely. Nevertheless, the concept of the Goldstone spinor is very appealing, and it is worth noticing that while there are general field theory arguments which forbid the existence of a Goldstone particle of higher spin, there are none for spin one-half [MAIS.71].

After all this analysis at the classical level, in the presence of a linear symmetry soft breaking term, the relevant to it renormalization program is readily to be advanced. In regard to that the unrenormalized but regularized version of the theory, shall once more be invoked. The Lagrangian reads:

$$L=L_0+L_\xi+L_m+L_g-cS \quad (V.131)$$

This time the fields S and F have non-vanishing vacuum expectation values, which at the tree approximation level are apparently seen to equal s and f respectively, and are expressed by equations (V.118) and (V.119). As a modification of these values in higher orders, is to be expected, they shall be denoted by τ_s and τ_f . They are to be determined as parts of the renormalization program.

This allows the definition of two shifted fields S' and F' such that:

$$S'=S-\tau_s \quad F'=F-\tau_f \quad \langle 0|S'|0\rangle=\langle 0|F'|0\rangle=0 \quad (V.132)$$

Likewise, the current becomes:

$$J^\mu = \gamma^\lambda \partial_\lambda (S' - \gamma_5 P) \gamma^\mu \chi - (F' + \gamma_5 G) \gamma^\mu \chi - \tau_F \gamma^\mu \chi \quad (V.133)$$

and it still satisfies the conservation equation (V.117). The charge associated with this current is the generator of supergauge transformations and therefore its action on the fields is known. The Ward identities can again be derived quite simply. Let $Z'[J]$ be the generating functional of the Green functions of the translated fields. Then combining with the basic relation already encountered in eq.(V.12):

$$Z[J] = \exp\{iW[J]\}$$

with the character of the generating functional for the connected Green functions $W[J]$ - which essentially expresses the interaction part of the action in the presence of properly chosen external sources - and the fact that the explicit symmetry breaking yields a current - which being only partially conserved, satisfies eq.(V.117) as opposed to $\partial_\mu J^\mu = 0$ immediately implying that unlike the case where the exact symmetry of the action is manifest, in this particular case, the variation $\delta Z'[J]$ is no longer a vanishing one, but as seen from eq.(V.117), satisfies:

$$\delta Z'[J] = -c \int d^4x \frac{\delta Z'[J]}{\delta \eta} \quad (V.134)$$

In addition since the explicit breaking term, has resulted in a shift in the fields S and F , it is to be expected that $\delta Z'[J]$ be of the same form as the left-hand side of

eq.(V.10), with two additional terms effected by the corresponding shifts. Combining that with eq.(V.134), results in:

$$\begin{aligned}
 & -i \int d^4x \frac{\delta Z'[J]}{\delta J_S} [\partial_\mu \bar{\eta}(x)] \gamma^\mu \\
 & + \tau_S \int d^4x Z'[J] [\partial_\mu \bar{\eta}(x)] \gamma^\mu \\
 & + i \int d^4x \frac{\delta Z'[J]}{\delta J_P} [\partial_\mu \bar{\eta}(x)] \gamma_5 \gamma^\mu \\
 & + i \int d^4x \frac{\delta Z'[J]}{\delta J_F} \bar{\eta}(x) \\
 & - \tau_F \int d^4x Z'[J] \bar{\eta}(x) \\
 & + i \int d^4x \frac{\delta Z'[J]}{\delta J_G} \bar{\eta}(x) \gamma_5 \\
 & - \int d^4x \frac{\delta Z'[J]}{\delta \eta} \{ \partial_\mu [J_F(x) - \gamma_5 J_G(x)] \gamma^\mu + J_S(x) + \gamma_5 J_P \} \\
 & = -c \int d^4x \frac{\delta Z'[J]}{\delta \eta}
 \end{aligned} \tag{V.135}$$

In this equation and in what follows, all functional derivatives with respect to J_S and J_F are essentially with respect to J_S and J_F .

Equation (V.135) contains all the Ward identities for the case of the broken supergauge symmetry. Again as was the case with the broken manifest symmetry of the Lagrangian, equation (V.135) can be written in terms of $W'[J]$. This

yields:

$$\begin{aligned}
 & \int d^4x \frac{\delta W'[J]}{\delta J_S} [\partial_\mu \bar{\eta}(x)] \gamma^\mu \\
 & + \tau_S \int d^4x [\partial_\mu \bar{\eta}(x)] \gamma^\mu \\
 & - \int d^4x \frac{\delta W'[J]}{\delta J_P} [\partial_\mu \bar{\eta}(x)] \gamma_5 \gamma^\mu \\
 & - i \int d^4x \frac{\delta W'[J]}{\delta J_F} \bar{\eta}(x) \\
 & - \tau_F \int d^4x \bar{\eta}(x) - i \int d^4x \frac{\delta W'[J]}{\delta J_G} \bar{\eta}(x) \gamma_5 \\
 & - \int d^4x \frac{\delta W'[J]}{\delta \eta} \{ \partial_\mu [J_F(x) - \gamma_5 J_G(x)] \gamma^\mu + J_S(x) + \gamma_5 J_P \} \\
 & = -c \int d^4x \frac{\delta W'[J]}{\delta \eta}
 \end{aligned} \tag{V.136}$$

Similarly, through the Legendre transformation introduced in (V.13), for the generating functional of proper vertices it holds:

$$\begin{aligned}
 & i \int d^4x \frac{\delta \Gamma'[R]}{\delta R_S} R_x \\
 & + i \int d^4x \frac{\delta \Gamma'[R]}{\delta R_P} \gamma_5 R_x \\
 & + i \int d^4x \frac{\delta \Gamma'[R]}{\delta R_F} \gamma^\mu \partial_\mu R_x \\
 & + i \int d^4x \frac{\delta \Gamma'[R]}{\delta R_G} \gamma_5 \gamma^\mu \partial_\mu R_x
 \end{aligned}$$

$$\begin{aligned}
& + \int d^4x (R_F + \tau_F + \gamma_5 R_G - \partial_\mu R_S \gamma^\mu - \partial_\mu R_P \gamma_5 \gamma^\mu) \frac{\delta \Gamma'[R]}{\delta R_x} \\
& = -ic \int d^4x R_x \quad (V.137)
\end{aligned}$$

It is stressed once more, at this point, that since the functional $\Gamma'[R]$, generalizes the classical action to higher orders in perturbation theory, (whence the term: "effective action"), it is bound to be subject to the same transformation properties as the Lagrangian. As a result equation (V.137) could have been laid out directly without any regard to the previous steps.

From this point, the renormalization program is identical to that of the symmetric theory. It commences again by tracking down the primitively divergent vertices to which appropriate counterterms shall have to be assigned. Subsequently, upon deriving the Ward identities through eq.(V.137), it can be proved that this assignment can be made in a way to respect them. In what follows, simply in order to demonstrate the relevant procedure, the derivation of a couple of them, is to be cited.

Taking the functional derivative of both sides of (V.137) with respect to $R_x(x')$, results in:

$$\begin{aligned}
& i \int d^4x \delta(x-x') \frac{\delta \Gamma'[R]}{\delta R_S} \\
& + i \int d^4x R_x \frac{\delta^2 \Gamma'[R]}{\delta R_x(x') \delta R_S(x)}
\end{aligned}$$

$$\begin{aligned}
& +i\gamma_5 \int d^4x \delta(x-x') \frac{\delta\Gamma'[R]}{\delta R_P} \\
& +i\gamma_5 \int d^4x R_x \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta R_P(x)} \\
& +i\gamma^\mu \int d^4x \partial_\mu \delta(x-x') \frac{\delta\Gamma'[R]}{\delta R_F} \\
& +i\gamma^\mu \int d^4x \partial_\mu R_x \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta R_F(x)} \\
& +i\gamma_5 \gamma^\mu \int d^4x \partial_\mu \delta(x-x') \frac{\delta\Gamma'[R]}{\delta R_G} \\
& +i\gamma_5 \gamma^\mu \int d^4x \partial_\mu R_x \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta R_G(x)} \\
& + \int d^4x R_F \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta \bar{R}_x(x)} \\
& + \tau_F \int d^4x \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta \bar{R}_x(x)} \\
& + \gamma_5 \int d^4x R_G \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta \bar{R}_x(x)} \\
& -i \int d^4x \partial_\mu R_S \gamma^\mu \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta \bar{R}_x(x)} \\
& -i \int d^4x \partial_\mu R_P \gamma_5 \gamma^\mu \frac{\delta^2\Gamma'[R]}{\delta R_x(x') \delta \bar{R}_x(x)} \\
& = -ic \int d^4x \delta(x-x')
\end{aligned}$$

where as before, the primes from the relevant derivatives involving S and F, have been omitted. Having accomplished that, differentiation with respect to $R_S(y)$ follows:

$$\begin{aligned}
& i \frac{\delta^2 \Gamma' [R]}{\delta R_S(y) \delta R_S(x')} \\
& + i \int d^4 x R_x \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta R_S(x)} \\
& + i \gamma_5 \frac{\delta^2 \Gamma' [R]}{\delta R_S(y) \delta R_P(x')} \\
& + i \gamma_5 \int d^4 x R_x \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta R_P(x)} \\
& - i \not{x} \frac{\delta^2 \Gamma' [R]}{\delta R_S(y) \delta R_F(x')} \\
& + i \int d^4 x \not{x} R_x \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta R_F(x)} \\
& - i \gamma_5 \not{x} \frac{\delta^2 \Gamma' [R]}{\delta R_S(y) \delta R_G(x')} \\
& + i \gamma_5 \int d^4 x \not{x} R_x \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta R_G(x)} \\
& + \int d^4 x R_{F'} \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta \bar{R}_x(x)} \\
& + \tau_F \int d^4 x \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta \bar{R}_x(x)} \\
& + \gamma_5 \int d^4 x R_G \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta \bar{R}_x(x)} \\
& - \int d^4 x \partial_\mu \delta(y-x) \gamma^\mu \frac{\delta^2 \Gamma' [R]}{\delta R_x(x') \delta \bar{R}_x(x)} \\
& - \int d^4 x \partial_\mu R_S \gamma^\mu \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_x(x') \delta \bar{R}_x(x)}
\end{aligned}$$

$$-\int d^4x \partial_\mu R_P \gamma_5 \gamma^\mu \frac{\delta^3 \Gamma' [R]}{\delta R_S(y) \delta R_X(x') \delta \bar{R}_X(x)} = 0$$

Then, replacing all derivatives with the corresponding proper functions renders the previous relation in the form:

$$\begin{aligned} & i\Gamma_{S'S}(y-x') + i \int d^4x R_X(x) \Gamma_{S'S'X}(y-x', x) + i\gamma_5 \Gamma_{S'P}(y-x') \\ & + i\gamma_5 \int d^4x R_X(x) \Gamma_{S'XP}(y-x', x) - i\cancel{\gamma} \Gamma_{S'F}(y-x') \\ & + i \int d^4x \cancel{\gamma} R(x) \Gamma_{S'XF'}(y-x', x) - i\gamma_5 \cancel{\gamma} \Gamma_{S'G}(y-x') \\ & + i\gamma_5 \int d^4x \cancel{\gamma} R_X(x) \Gamma_{S'XG}(y-x', x) + \int d^4x R_F(x) \Gamma_{S'XX}(y-x', x) \\ & + \tau_F \int d^4x \Gamma_{S'XX}(y-x', x) + \gamma_5 \int d^4x R_G(x) \Gamma_{S'XX}(y-x', x) \\ & + \cancel{\gamma} \Gamma_{XX}(x'-y) - \int d^4x \partial_\mu R_S(x) \gamma^\mu \Gamma_{S'XX}(y-x', x) \\ & - i \int d^4x \partial_\mu R_P(x) \gamma_5 \gamma^\mu \Gamma_{S'XX}(y-x', x) = 0 \end{aligned}$$

Subsequently, setting all classical fields to zero:

$$\begin{aligned} & i\Gamma_{S'S}(y-x') + i\gamma_5 \Gamma_{S'P}(y-x') - i\cancel{\gamma} \Gamma_{S'F}(y-x') - i\gamma_5 \cancel{\gamma} \Gamma_{S'G}(y-x') \\ & + \tau_F \int d^4x \Gamma_{S'XX}(y-x', x) + \cancel{\gamma} \Gamma_{XX} = 0 \end{aligned}$$

Now translation invariance entails momentum conservation, so

that:

$$G^{(N)}(x_1, \dots, x_N) = (2\pi)^4 \int d^4 p_1 \dots d^4 p_N \\ \cdot e^{i(p_1 x_1 + \dots + p_N x_N)} \tilde{G}(p_1, \dots, p_N) \delta^{(4)}(p_1 + \dots + p_N)$$

Applying that to the previous relation, yields its Fourier transformation:

$$\begin{aligned} & i(2\pi)^4 \int d^4 p_1 d^4 p_2 e^{i(p_1 x' + p_2 y)} \Gamma_{S'S'}(p_1, p_2) \delta^{(4)}(p_1 + p_2) \\ & + i\gamma_5 (2\pi)^4 \int d^4 p_1 d^4 p_2 e^{i(p_1 x' + p_2 y)} \Gamma_{S'P}(p_1, p_2) \delta^{(4)}(p_1 + p_2) \\ & - (2\pi)^4 \int d^4 p_1 d^4 p_2 (-i\not{p}_1)(-i\not{p}_2) e^{i(p_1 y + p_2 x')} \Gamma_{S'F}(p_1, p_2) \delta^{(4)}(p_1 + p_2) \\ & - i\gamma_5 (2\pi)^4 \int d^4 p_1 d^4 p_2 (-i\not{p}_1)(-i\not{p}_2) e^{i(p_1 y + p_2 x')} \Gamma_{S'G}(p_1, p_2) \delta^{(4)}(p_1 + p_2) \\ & + (2\pi)^4 \tau_F \int d^4 p_1 d^4 p_2 d^4 p_3 d^4 x e^{i(p_1 y + p_2 x' + p_3 x)} \\ & \cdot \Gamma_{S'XX}(p_1, p_2, p_3) \delta^{(4)}(p_1 + p_2 + p_3) \\ & + (2\pi)^4 \int d^4 p_1 d^4 p_2 (-i\not{p}_1)(-i\not{p}_2) e^{i(p_1 y + p_2 x')} \Gamma_{XX}(p_1, p_2) \delta^{(4)}(p_1 + p_2) = 0 \end{aligned}$$

The fifth integral involving $\Gamma_{S'XX}(p_1, p_2, p_3)$ is to be manipulated in the following manner:

$$\int d^4 p_1 d^4 p_2 d^4 p_3 d^4 x e^{i(p_1 y + p_2 x' + p_3 x)} \Gamma_{S'XX}(p_1, p_2, p_3) \delta^{(4)}(p_1 + p_2 + p_3)$$

$$\begin{aligned}
&= \int d^4 p_3 d^4 p_2 d^4 x e^{i[p_3 x + p_2 x' - (p_3 + p_2)y]} \Gamma_{S'XX}(p_2, p_3) \\
&= \int d^4 p_3 d^4 p_2 e^{i[p_2(x'-y) - p_3 y]} \Gamma_{S'XX}(p_2, p_3) \delta^{(4)}(p_3) \\
&= \int d^4 p_2 e^{-i p_2(y-x')} \Gamma_{S'XX}(p_2, p_3=0)
\end{aligned}$$

Likewise, integrating out all δ -functions and replacing everything in the previous relation results in the final expression:

$$\begin{aligned}
&\Gamma_{S'S'}(p^2) + i\gamma_5 \Gamma_{S'P}(p^2) - p^2 \Gamma_{S'F}(p^2) \\
&- i\gamma_5 p^2 \Gamma_{S'G}(p^2) + \tau_F \Gamma_{S'XX}(p_i) + p^2 \Gamma_{XX}(p^2) = 0
\end{aligned} \tag{V.138}$$

which is one of the Ward identities the theory entails. Then setting $p=0$, in accordance with the requirement that the subtractions be performed at the origin in momentum space, renders the previous expression in the final form:

$$\Gamma_{S'S'}(p^2=0) + i\gamma_5 \Gamma_{S'P}(p^2=0) = -\tau_F \Gamma_{S'XX}(p_1=p_2=0)$$

Obviously, the term involving $\Gamma_{S'P}$, which itself represents the connected part of the S'P Feynman diagram, drops out of this equation, being utterly precluded from the structure of the shifted Lagrangian. Hence finally:

$$\Gamma_{S'S'}(p^2=0) = -\tau_F \Gamma_{S'XX}(p_i=0) \tag{V.139}$$

Likewise, taking two successive derivatives of (V.138) with respect to $R_x(x')$ and $R_S(y)$, in the manner previously performed, and differentiating once more with respect to R_S , subsequently letting the classical fields go to zero, results in:

$$\begin{aligned}
 & i \frac{\delta^3 \Gamma'[R]}{\delta R_S(y) \delta R_S(x') \delta R_S(x)} + i \gamma_5 \frac{\delta^3 \Gamma'[R]}{\delta R_S(y) \delta R_P(x') \delta R_S(x)} \\
 & - i \not{x} \frac{\delta^3 \Gamma'[R]}{\delta R_S(y) \delta R_X(x') \delta R_S(x)} - i \gamma_5 \not{x} \frac{\delta^3 \Gamma'[R]}{\delta R_S(y) \delta R_S(x') \delta R_G(x)} \\
 & + \tau_F \int d^4 x \frac{\delta^4 \Gamma'[R]}{\delta R_S(y') \delta R_S(y) \delta R_X(x') \delta \bar{R}_X(x)} \\
 & - i \int d^4 x \not{x} \delta(x-y) \frac{\delta^3 \Gamma'[R]}{\delta R_S(y) \delta R_X(x') \delta \bar{R}_X(x)} = 0
 \end{aligned}$$

Then Fourier transforming all terms in the previous equation, as before, yields:

$$\begin{aligned}
 & i \Gamma_{S'S'S'}(p^2) - i \not{x} \Gamma_{S'XS'}(p_i) - i \gamma_5 \Gamma_{S'S'S'}(p_i) \\
 & + \tau_F \Gamma_{S'S'XX}(p_i) - \not{x} \Gamma_{S'XX}(p_i) = 0
 \end{aligned} \tag{V.140}$$

being aware of the fact that the structure of the shifted Lagrangian precludes the inclusion of the term $i \gamma_5 \Gamma_{S'PS'}(p^2)$ in equation (V.140).

Finally, setting all momenta to zero, renders the final form:

$$\Gamma_{S'S'S'}(p_i=0) = \tau_F \Gamma_{S'S'_{XX}}(p_i=0) \quad (V.141)$$

As was mentioned before, once the Ward identities have been derived the renormalization program is identical to that of the symmetric theory. In terms of the renormalized fields, the current in eq.(V.133) turns out to be:

$$J^\mu = Z[\gamma^\lambda \partial_\lambda (S'_R - \gamma_5 P_R) \gamma^\mu \chi_R - (F'_R + \gamma_5 G_R) \gamma^\mu \chi_R - \tau_{F_R} \gamma^\mu \chi_R] \quad (V.142)$$

with:

$$\partial_\mu J^\mu = c_R \chi_R \quad (V.143)$$

where obviously:

$$\tau_{F_R} = Z^{-1/2} \tau_F \text{ and } c_R = Z^{1/2} c$$

Some examples of the Ward identities, expressed in terms of the renormalized proper functions, including the ones derived above in (V.139) and (V.141), are:

$$\tau_{F_R} \Gamma_{XX}^{(2)}(p^2=0)_R = c_R \quad (V.144a)$$

$$\Gamma_{S'S'}(p^2=0)_R = \tau_{F_R} \Gamma_{XXS'}(p_i=0)_R = -\Gamma_{PP}(p^2=0)_R \quad (V.144b)$$

$$-\Gamma_{S'S'}(p^2=0)_R = \Gamma_{XX}^{(1)}(p^2=0)_R + O(g^2) = \Gamma_{F'F'}(p^2=0)_R + O(g^2) \quad (V.144c)$$

$$\Gamma_{XX}^{(2)}(p^2=0)_R = \Gamma_{F'S'}(p^2=0)_R + O(g^2) \quad (V.144d)$$

$$\Gamma_{PP}'(p^2=0)_R + O(g^2) = \Gamma_{S'S'}'(p^2=0)_R + O(g^2) \quad (V.144e)$$

$$\Gamma_{S'S'S'}(p_i=0)_R = O(g^2) \quad (V.144f)$$

where $O(g^2)$ stands for terms which are not renormalization parts of the theory. They vanish in the tree approximation, and their values in higher orders are finite and calculable. For example the last equation (V.144f) reads as in (V.141). Now since the superficial degree of divergence as expressed by eq.(V.31) was derived through manipulations of the interaction part of the Lagrangian, that the symmetry breaking leaves intact - as seen from eq.(V.121) - it is expected to remain in the same form after that breaking. Consequently, the proper function $\Gamma_{S'S'S'}$ in (V.141), graphically expressed in the one loop level as:



fig. V.16

is convergent as having a superficial degree of divergence equal to -1, whence the finite part on the right-hand side of eq.(V.144f).

It is obvious that the only new constant which needs to be introduced can be chosen to be c_R . For example, although $\Gamma_{S'S'}(p^2=0)$ - which was not a renormalization part in the

exact symmetry case - does not vanish, it is expressed in terms of proper functions, which are always renormalization parts of the theory. The same for $\Gamma_{S'S'S'}(p_i=0)$, which is finite and calculable.

This, concludes the entire discussion on the quantum field theoretical properties of the N=1 Wess - Zumino model.

VI. THE PHENOMENOLOGY OF SUPERSYMMETRY

A. General considerations

A description of the most compelling motivations for accepting SUSY as the most likely model of physics, beyond the standard model, has already been attempted. In what follows some phenomenological implications and experimental advances in this field, will be furnished.

SUSY is a symmetry which relates fermionic and bosonic degrees of freedom. In addition - as was already stressed in the first chapter - the total number of degrees of fermionic and bosonic degrees of freedom, must be equal. As an example of this statement, a quark which relates, through SUSY, to two scalar quarks shall be taken into consideration.

The counting is to be performed as follows: The degrees of freedom are counted conveniently by regarding separately the left-handed and right-handed quark chiralities q_L and q_R . Accordingly, there will be two complex scalar quarks $\tilde{q}_L$ and $\tilde{q}_R$. Since q_L and $\tilde{q}_L$ each have two degrees of freedom, the counting matches. An equivalent way of expressing the same fact, is to consider the Dirac spinor q as consisting of a two component quark and its antiquark, which corresponds in number to two scalar quarks and their scalar antiquarks. The minimal set of new supersymmetric particles needed, constituting the supersymmetric particle spectrum may now be put forward [HAB.85] [ELL.83].

TABLE

Normal particles	Supersymmetric partners				Mass eigenstates with specific couplings	
	Weak interaction eigenstates		Mass eigenstates			
	Symbol	Name	Symbol	Name	Symbol	Name
$q=u,d,s, c,b,t$	$\bar{q}_L, \bar{q}_R$	scalar-quark	$\bar{q}_1, \bar{q}_2$	scalar-quark		
$\ell=e,\mu,\tau$	$\bar{\ell}_L, \bar{\ell}_R$	scalar-lepton	$\bar{\ell}_1, \bar{\ell}_2$	scalar-lepton		
$\nu=\nu_e, \nu_\mu, \nu_\tau$	$\bar{\nu}$	scalar-neutrino	$\bar{\nu}$	scalar-neutrino		
g	g	gluino	g	gluino		
$\tilde{W}^\pm$	$\tilde{W}^\pm$	wino			$\tilde{W}^\pm$	wino
$H_1^\pm$	$\tilde{H}_1^\pm$	higgsino	$\tilde{X}_{1,2}^\pm$	charginos	$\tilde{h}^\pm$	higgsino
$H_2^\pm$	$\tilde{H}_2^\pm$	higgsino			$\tilde{\omega}_1, \tilde{\omega}_2$	wiggsino
γ	$\tilde{\gamma}$	photino			$\tilde{\gamma}$	photino
Z^0	$\tilde{Z}^0$	zino			$\tilde{Z}$	zino
H_1^0	$\tilde{H}_1^0$	higgsino	$\tilde{X}_1^0$	neutralinos	$\tilde{h}_1, \tilde{h}_2$	higgsino
H_2^0	$\tilde{H}_2^0$	higgsino			$\tilde{\tau}_1, \tilde{\tau}_2$	ziggsino
$\begin{pmatrix} W^3 \\ B \end{pmatrix}$	$\begin{pmatrix} \tilde{W}^3 \\ \tilde{B} \end{pmatrix}$	$\begin{pmatrix} \text{wino} \\ \text{bino} \end{pmatrix}$			$\begin{pmatrix} \tilde{W}^3 \\ \tilde{B} \end{pmatrix}$	$\begin{pmatrix} \text{wino} \\ \text{bino} \end{pmatrix}$

This table is based on the standard nomenclature used for supersymmetric partners. Generally, for the partners of bosons, mixing occurs among the weak eigenstates, suggesting the need for different symbols and names between them and the mass eigenstates. Couplings of the weak eigenstates are determined by the theory, but couplings of mass eigenstates depend on the amount of mixing. Z^0 and γ , are linear combinations of W^3 and B , with the same combination of $\tilde{W}^3$, $\tilde{B}$ giving $\tilde{Z}^0$, and $\tilde{\gamma}$. If SUSY is a spontaneously broken global symmetry, there is also a Goldstone fermion, called the Goldstino ($\tilde{G}$). In Supergravity theories (SUGRA), it is absorbed by a gravitino ($\tilde{g}$), which in effect acquires a mass.

The following properties of supersymmetric particles are noteworthy. With the exception of spin the supersymmetric particles states are described by the same quantum numbers as those describing ordinary particle states. In addition, since in SUSY not all Higgs-fermion interaction terms present in the conventional standard model are allowed, at least two Higgs doublets are needed in order to generate masses for all fermions. Finally, SUSY restoration at appropriate energies, poses a problem that any sensible theory would have to settle, that relates to the question why all supersymmetric particle masses are lifted above the corresponding ones of their partners, high enough to elude experimental detection.

The supersymmetric spectrum presented above, contains neutral massive Majorana fermions such as the photino. As a result there can be fermion number violation as exemplified by such reactions as $e^- + e^- \rightarrow \tilde{e}^- + \tilde{e}^-$ which goes through a t-channel exchange of a Majorana photino. This would not however be detected experimentally as such, since the scalar electrons would quickly decay to electrons - through a mechanism soon to be analyzed - so that no fermion number violation would eventually seem to occur.

If the photino and gluino were massless, then a generalization of fermion number conservation could be defined. In the context of SUSY the relevant symmetry, is a continuous symmetry of a global character called R-symmetry. The gluino however does not seem to be massless, in which case this continuous R-symmetry, must be broken.

It turns out that despite this breaking, a discrete R-symmetry, always remains. Ordinary particles are assigned an R-parity of +1, whereas the supersymmetric partners, have an R-parity of -1. The first important consequence of that, is that supersymmetric particles must be produced in pairs. This is because in laboratory experiments the initial state cannot contain any supersymmetric particles so as a result the final state can only contain an even number of such states. Further, a lightest supersymmetric particle must exist. On R-parity conservation grounds, such a particle could not decay into only non supersymmetric particles.

In dealing with spontaneously broken global supersymmetry,

the lightest supersymmetric particle is the goldstino, since it happens to be massless. However in SUGRA models the goldstino is absorbed - as already mentioned - into a massive gravitino which in most recent models, is not the lightest supersymmetric particle. In what follows, the underlying assumption is that the photino is the lightest supersymmetric particle, unless otherwise explicitly stated. Reference will be made later on, regarding the scalar neutrino, which constitutes a good possible candidate for a light supersymmetric particle as well.

What shall be ignored - for brevity's shake - is possible complications effected by mixing with other neutral Majorana fermions which would cause the lightest state to be a mixed state of photino, zino and higgsino. The third important implication effected by the R-parity conservation, is that the production cross-section for any light supersymmetric particle, is most likely of weak interaction size. Consider for example the processes:

$$e^+e^- \rightarrow \tilde{\gamma}\tilde{\gamma}$$

This occurs via a t-channel exchange of a scalar electron. Since - as shall be discussed - the absence of any experimental evidence of its existence implies that the scalar electron is heavy (definetely $\tilde{M}_e \sim O(22 \text{ GeV})$), the virtual $\tilde{e}$ exchange supresses the $\tilde{\gamma}$ production cross section. Likewise the $\tilde{\gamma}$ interaction cross section is of weak interaction size because it can only interact with matter via the exchange of a heavy scalar quark. As a result the $\tilde{\gamma}$

will behave like a neutrino. This observation which would also hold true for the lightest supersymmetric particle, is the basis for most of the experimental signatures of supersymmetry.

It has been assumed that the R-parity is an exact symmetry. If it were broken then the lightest supersymmetric particle would be unstable. Almost certainly such a situation would also require the violation of lepton number (e.g. $\tilde{\gamma} \rightarrow \gamma + \nu$)

The whole discussion so far reveals that there is one basic experimental signature which could distinguish SUSY from the standard model physics. Specifically once supersymmetric particles are produced, they will eventually decay into photinos which like neutrinos, will simply elude experimental detection (except for neutrino and photino detectors). However neutrinos are almost always produced in association with a charged lepton, whereas the photino is often produced with no hard charged leptons present. Consequently there is the possibility of finding events where substantial amounts of energy and momentum are lost, but no lepton is detected. This constitutes the basis of almost all experimental searches for supersymmetric phenomena.

As has been argued so far, the gauge hierarchy problem seems to strongly imply that new physics will enter in the 100GeV to 1TeV region. That does not however definitely suggest that this new physics will be at all related to

supersymmetry. The nonexistence of any experimental evidence for supersymmetry so far, might be taken to imply that this theory is probably relevant only at the Planck scale. Nevertheless, in searching for new phenomena it is important to consider how new physics can emerge from Standard Model backgrounds. In a sense supersymmetry is serving as a model for new physical phenomena anticipated in the next energy regime. Since SUSY is essentially a weak coupling theory, it allows for detailed calculations of a whole assortment of possible new physics. In each case, detailed study of standard model backgrounds, subsequently reveals where new physics can be uncovered.

It is stressed at this point, that more or less the ensuing discussion is model independent. Most of the model dependence in discussions of supersymmetric models, arises because the origin of SUSY breaking is not understood. This shows up in a model dependence of masses of superpartners. However all limits will be laid out as functions of masses so that model dependence is automatically precluded, allowing of course for the case in which mass matrices cause mixing of weak eigenstates, consequently inducing some relevant model dependence.

Furthermore the ensuing analysis shall adhere in the phenomenological implications of the $N=1$ model, already discussed so far from a formal point of view. This is predicated by the fact that the effective low energy theory compatible with the $SU_L(2) \otimes U_Y(1)$ electroweak theory must be

an $N=1$ SUSY. Theories with $N>1$ result in unacceptable - from the Standard Model's point of view $SU(2)$ multiplets consisting of both left and right handed fermions connected by a supersymmetric transformation. Hence consideration of - only - the phenomenology of the $N=1$ SUSY, entails no loss of generality.

Another point worth mentioning, is that - generally speaking - a new part of physics, could be detected by finding a new particle or a new interaction. At accelerators new particles are much easier to detect than new interactions, which themselves allow by nature many diverse interpretations, even if found.

The final elucidation to be given, before turning to a survey of the supersymmetric particle spectrum, concerns the supersymmetric model itself. Any discussion of production and decay of supersymmetric particles, would necessitate the full interaction Lagrangian. However it proves quite convenient to construct all necessary vertices by simply taking the Standard Model three particle vertices and replacing ordinary particles by their superpartners in pairs. e.g.:

$$e^+ e^- \gamma \rightarrow e^+ e^- \gamma + e^+ e^- \gamma$$

(i) Photino interactions

The detection of supersymmetric partners is experimentally based on either: (a) detecting the interaction of a light superpartner at the end of a decay

chain, or (b) missing momentum transfer p_T because such a particle escapes a detector or both. As has already been mentioned, the missing p_T can be distinguished from that of a neutrino, because no charged lepton accompanies the photino.

The incoming photino interacts by exciting or exchanging a quark. The quarks and gluons are constituents inside the hadron. e.g.:

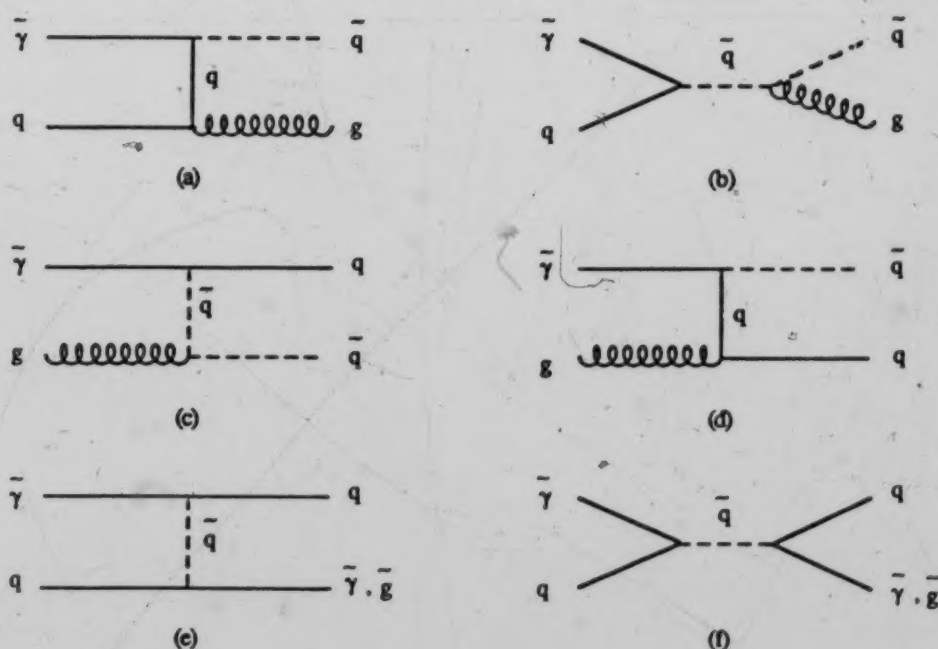


fig. VI.1

The $\bar{q}$ can decay to $\bar{q}g$, if $M_{\bar{q}}^2$ is less than xs - with $x \in (0,1)$ and s being the centre of mass (energy)² for the relevant decay process - otherwise $q\bar{q}$ is available. If the

$\tilde{g}q$ mode is allowed, it will dominate because the coupling is stronger. The full interaction cross-section $\sigma_{int}^{\tilde{\gamma}}$ has been calculated to be proportional to the squared energy s associated with the photino-hadron system. If $E_{\tilde{\gamma}}$ is the photino lab energy, then: $s=2E_{\tilde{\gamma}}m_{proton}$ [FAY.79] [KAN.82]. The relevant signature for the photino interactions consists of an event with no hard charged lepton and very little missing energy or $p_{\perp}$ (in beam dump detectors), so it is a neutral current candidate, but one with different characteristics from neutrino events. Specifically, its distributions are those for scalar currents, and there is less missing energy. The signature in a collider is the $p_{\perp}$ carried away, with no associated hard charged lepton.

If instead of the photino, the scattering process involves a higgsino, or a goldstino, or a scalar neutrino, there are some differences but the main points of the analysis above remain the same [FAY.79].

The aforementioned remarks, suffice for an analysis of the main signatures relevant to the other supersymmetric particles.

(ii) Scalar leptons

There are two scalar leptons, denoted by $\tilde{l}_L$ and $\tilde{l}_R$ for each generation of leptons, l . That is, there is one ($\tilde{l}_L$), associated with the left handed lepton l_L (which is in a weak $SU(2)$ doublet), and a different one

($\tilde{l}_R$), associated with the right-handed lepton l_R , (which is in a weak SU(2) singlet). In general $\tilde{l}_L$ and $\tilde{l}_R$ are expected to mix and the mass eigenstates will be model dependent linear combinations of these states. For simplicity, the analysis shall relate to the $\tilde{l}_R$, and $\tilde{l}_L$. When not important, the L and R subscript will be dropped and use will be made of $\tilde{l}$.

If the scalar leptons are to be found they shall have to be produced in e^+e^- colliders and reference is made here to their signatures at these machines.

B. The decay of the Scalar lepton

If photinos are light, the scalar lepton decays rapidly with essentially 100% branching ratio, into:

$$\tilde{l} \rightarrow l \gamma$$

and

$$\tilde{l} \rightarrow l \tilde{\gamma}$$

with [FAR.80]:

$$\Gamma(\tilde{l} \rightarrow l \tilde{\gamma}) = \alpha (\tilde{M}_L^2 - m_L^2 - \tilde{M}_\gamma^2) p_1 / \tilde{M}_1^2 \quad (\text{VI.a})$$

where $p_1 = [\lambda(\tilde{M}_1^2, p_1^2, \tilde{M}_\gamma^2) / 2\tilde{M}_1]^1/2$ is the lepton momentum in the rest frame of the scalar lepton. The photino interacts too weakly to be detected in a normal collider detector, so it escapes. Hence in the $\tilde{l}$ rest frame, if the mass of the lepton is neglected, at least half the energy of the event

escapes. If the photino is heavier than the scalar lepton, then the scalar lepton decay depends on what the lightest supersymmetric particle is. It could for instance be assumed to be stable (despite limits and astrophysical constraints on the existence of charged particles). If the scalar neutrinos were lighter, then the β -decay type process $\tilde{l} \rightarrow l \nu$ occurs at a weak interaction rate or faster (depending on the wino which mediates the decay). The analysis shall adhere to the assumption that $\tilde{l} \gamma$ is the dominant decay mode, bearing in mind that neither theory nor experiment preclude the eventuality of a different decay mode.

C. The production of scalar leptons

It would appear that, since charged scalar leptons, interact electromagnetically, e^+e^- physics is the most appropriate environment to look for them.

The charged scalar leptons are ~~straightforward~~ to produce in pairs via $e^+e^- \rightarrow \tilde{l}_L \tilde{l}_L$. They are produced by direct channel γ and Z^0 as for any charged scalar. For $e^+e^- \rightarrow \tilde{e}^+ \tilde{e}^-$, it should also be included t-channel photino and zino exchange. One photon exchange predicts $1/4\beta^3$ units of R, each for $\tilde{l}_L$, $\tilde{l}_R$ and a $\sin^2\theta$ production distribution. In this particular case a rise in R can only be observed far above threshold. However, since half of the energy of the event escapes to unobserved photinos, it is possible to identify the occurrence of scalar lepton production. This indicates that assuming a light photino, the mass limits experimentally

imposed on the mass of the scalar lepton, should be close to the electron beam energy. All present detectors have results on limits on scalar lepton masses based on the non observation of $e^+e^- \rightarrow \tilde{l}^+\tilde{l}^-$.

At larger center of mass energies $\sqrt{s}$, Z^0 exchange will become important. The differential cross section still exhibits a $\sin^2\theta$ distribution.

For $e^+e^- \rightarrow \tilde{e}^+\tilde{e}^-$, the t-channel photino exchange becomes important. Again for larger centre of mass energies the Z^0 and zino contributions will become important, and an exact form for the cross section, including interferences will be required.

Two methods, sensitive to the scalar electron masses larger than the beam energy, are worth mentioning. Both of them assume light photinos. The first method, suggested by Gaillard, Hall, and Hinchliffe [GAIL.82], involves associated production of the scalar electron and photino via $e^+e^- \rightarrow e^+\tilde{e}^-\tilde{\gamma}$ as shown:

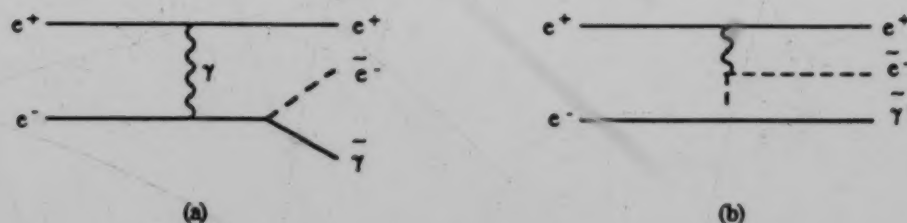


fig. VI.2

Assuming that $\tilde{e} \rightarrow e^+\tilde{\gamma}$, one is effectively searching for $e^+e^- \rightarrow e^+\tilde{e}^-\tilde{\gamma}\tilde{\gamma}$. The relevant signal consists of an acoplanar e^+e^-

pair and substantial missing energy in the event. The advantage of this process is that for light photinos the result is sensitive to scalar electron masses larger than the beam energy.

The second method for obtaining more severe mass limits for the scalar leptons was suggested by Griffolds [GRI.82], Fayet [FAY.82] and by Ellis and Hagelin [ELL.82]. They proposed a search for $e^+e^- \rightarrow \tilde{\gamma}\tilde{\gamma}\gamma$, which is a radiative correction to e^+e^- annihilation into photino pairs. The advantage of the extra photon is that it serves to tag the event. Otherwise, since the photinos are assumed to escape, the effect would be invisible. (This is obviously a variation of the neutrino counting experiments). In the case of photinos, the cross section is sensitive to the scalar electron mass because it appears in the propagator

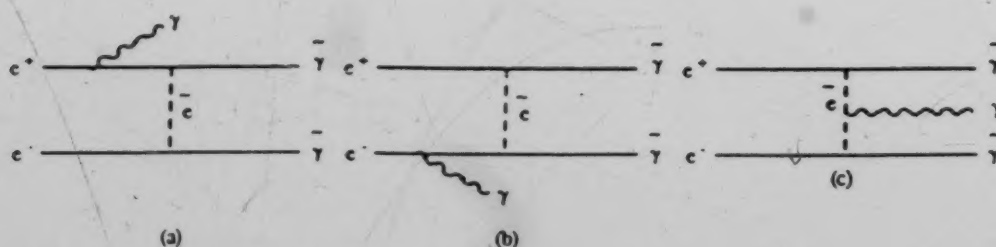


fig. VI.3

Apparently, measurement of this process, could be sensitive - if the photino is light - to scalar electron masses which are as large as 60 GeV. Of course, observation of $e^+e^- \rightarrow \gamma + X$, where X is invisible, would not necessarily imply SUSY. However, observation of such a signal, beyond that required

for three families of neutrinos, would be extraordinarily important and would require some new physics whose nature would have to be determined separately.

VII. AND BEYOND....

A. Concluding Remarks

The entire discussion, has placed emphasis on the - historically - first renormalizable supersymmetric model. The subsequent development of Supersymmetry, despite impressive successes, failed to eventually result in a unified model consistent with Quantum Field Theory.

On the experimental side, as of the present moment no unambiguous observation advocating Supersymmetry, has been reported.

It still remains an open question, whether the celebrated Superstring Theories - the closed version of which, allows for Supersymmetry - will eventually constitute the decisive step, in the search for the ultimate unified field theory. That epic quest of the human mind for the "Holy Grail".

[MAL.]

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APPENDIX--Representations of the Lorentz group

The algebra of the Lorentz group is normally expressed in terms of six Hermitian generators, arranged into an antisymmetric second-rank tensor $M_{\mu\nu}$. However, it can also be expressed in terms of six non-Hermitian generators $\bar{J}_\pm$ with:

$$J_\pm^1 \equiv \frac{1}{2}(M^{23} \pm iM^{01}) \text{ and cyclic } (1 \rightarrow 2 \rightarrow 3 \rightarrow 1) \quad (\text{A.1})$$

and subsequently assumes the form:

$$[J_\pm^1, J_\pm^2] = iJ_\pm^3 \text{ and cyclic,}$$

$$[\bar{J}_\pm, \bar{J}_\mp] = 0 \quad (\text{A.2})$$

Finite, dimensional irreducible representations, are classified by pairs of half-integer numbers (j_+, j_-) received from the the eigenvalues $j_\pm(j_\pm+1)$ of the two Casimir operators $(\bar{J}_\pm)^2$. The algebra is not that of $SU(2) \times SU(2)$ because the J 's are not Hermitian:

$$(\bar{J}_\pm)^\dagger = \bar{J}_\mp \quad (\text{A.3})$$

Correspondingly, whereas the J 's can be represented by finite dimensional Hermitian matrices, the $M_{\mu\nu}$'s cannot. The representation $(\frac{1}{2}, 0)$, for instance, has:

$$r(\bar{J}_+) = \frac{1}{2}\bar{\sigma}; \quad r(\bar{J}_-) = 0 \quad (\text{A.4})$$

($\bar{\sigma}$ are the familiar Pauli matrices) and hence:

$$r(M^{0i}) \equiv \frac{1}{2}\bar{\sigma}^{0i} = -\frac{1}{2}i\sigma^i; \quad r(M^{12}) \equiv \frac{1}{2}\bar{\sigma}^{12} = \frac{1}{2}\sigma^3 \text{ and cyclic} \quad (\text{A.5})$$

The group generated by these matrices is $SL(2, c)$. For the representation $(0, \frac{1}{2})$ it holds:

$$r(\bar{J}_+) = 0; \quad r(\bar{J}_-) = \frac{1}{2}\bar{\sigma} \quad (\text{A.6})$$

and hence:

$$r(M^{0i}) \equiv \frac{1}{2}\bar{\sigma}^{0i} = \frac{1}{2}i\sigma^i; \quad r(M^{12}) \equiv \frac{1}{2}\bar{\sigma}^{12} = \frac{1}{2}\sigma^3 \text{ and cyclic} \quad (\text{A.7})$$

As it is obvious, in either case, the generators $\bar{L}$ of the proper rotations are represented by $\frac{1}{2}\bar{\sigma}$.

Since $\bar{\sigma}^{\mu\nu} = (\sigma^{\mu\nu})^\dagger$, it would be appropriate to assign "index types" $(\sigma^{\mu\nu})_{\alpha\beta}$ and $(\bar{\sigma}^{\mu\nu})^{\alpha\beta}$ to these representations.